MATH 1011 : MULTIVARIABLE CALCULUS

INTRODUCTION : what is multivariable calculus?

Calculus

limits, continuity, differentiation and integration of functions $: \mathbb{R} \to \mathbb{R}$

Variables

$\mathbb{R}$ real numbers

$\mathbb{R}^2$ = set of pairs x, y such that x, y are elements of $\mathbb{R}$

$= \{ (x,y) : x, y \in \mathbb{R} \}$ notation: $\underset{\sim}{x} = (x,y)$ or (x_1, x_2)

$\mathbb{R}^3 = \{ (x,y,z) : x, y, z \in \mathbb{R} \}$ $\underset{\sim}{x} = (x, y, z)$ or (x_1, x_2, x_3)

triples

$\vdots$

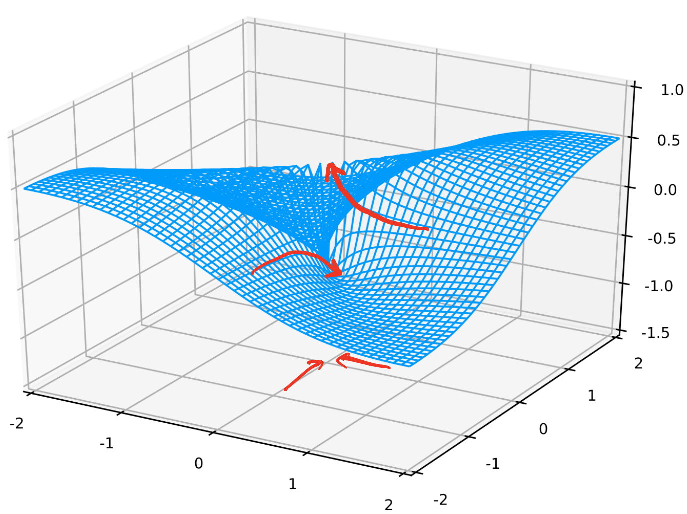
$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_1, x_2 \ldots x_n \in \mathbb{R} \}$

n-tuples

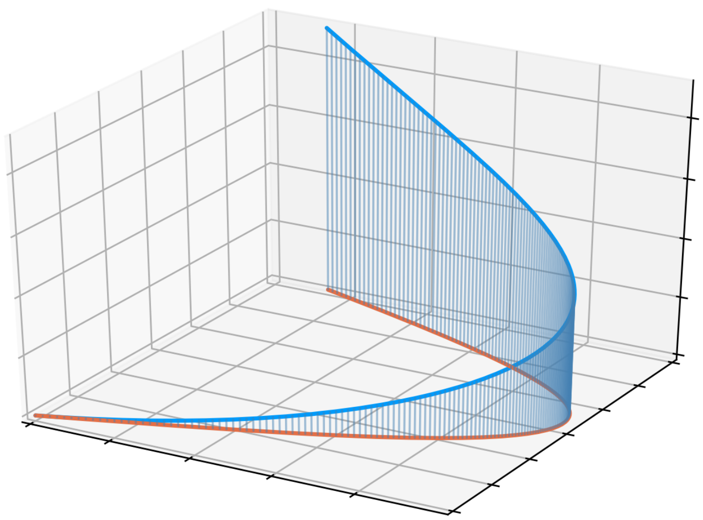
Multivariable calculus

extend limits, continuity, differentiation and integration to functions $\mathbb{R}^2 \to \mathbb{R}$, $\mathbb{R} \to \mathbb{R}^3$, $\mathbb{R}^3 \to \mathbb{R}^3$, etc.

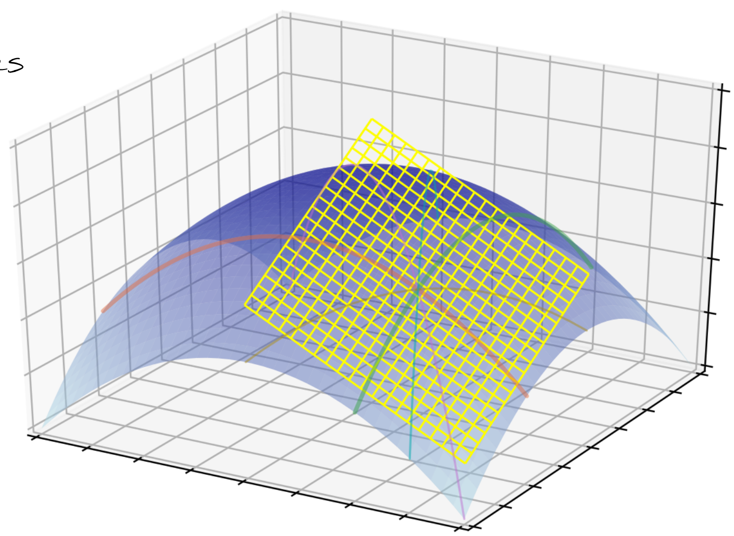
eg: limits of functions of two variables



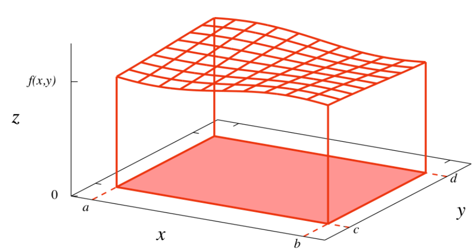
line integrals



tangent planes



double integrals



and other things too...

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) : x_1, x_2 \dots x_n \in \mathbb{R} \}$$

n-tuples

$\mathbb{R}^n$ is a vector space:

zero vector $\quad \underset{\sim}{0} = (0, 0, \dots, 0)$

addition $\quad \underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n \quad \underset{\sim}{u} = (u_1, u_2, \dots, u_n) \quad \underset{\sim}{v} = (v_1, v_2, \dots, v_n)$

$$\underset{\sim}{u} + \underset{\sim}{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

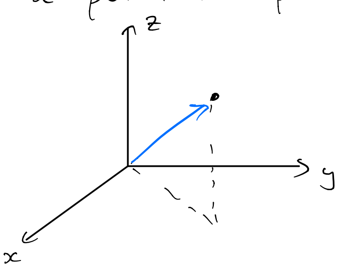
scalar multiplication $\quad \alpha \in \mathbb{R}$

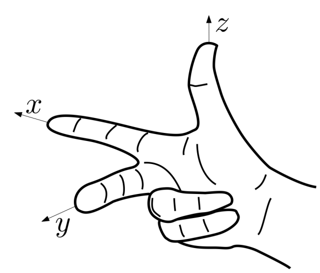
$$\alpha \underset{\sim}{u} = (\alpha u_1, \alpha u_2, \dots, \alpha u_n)$$

an element $\underset{\sim}{x} \in \mathbb{R}^n$ might represent:

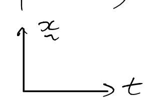
- coordinates of a point in space $(n = 3)$

eg: $(1,1,1)$





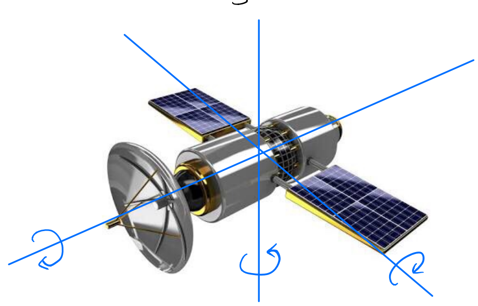
- or a vector (instructions for getting to a point)
- coordinates of a point in spacetime



$(n=4)$

- location and orientation of a satellite $\quad (n = 6)$

location: 3, orientation: 3 $\quad \mathbb{R}^6$



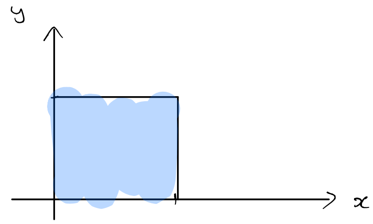
Note: since $\mathbb{R}^n$ is a vector space, its elements are called vectors, even if they are being used to represent points.

subsets of $\mathbb{R}^n$ (examples)

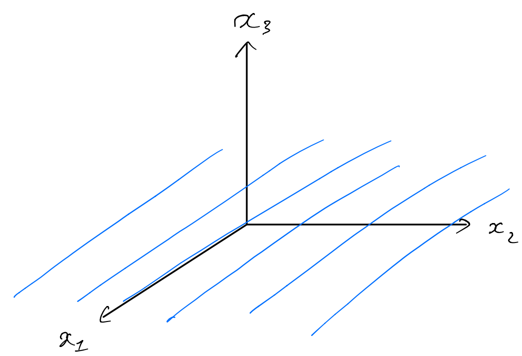
$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ $\qquad A \subseteq \mathbb{R}^n$

A is a subset of $\mathbb{R}^n$

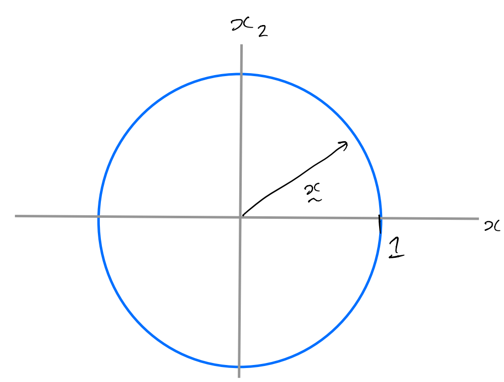
graphically



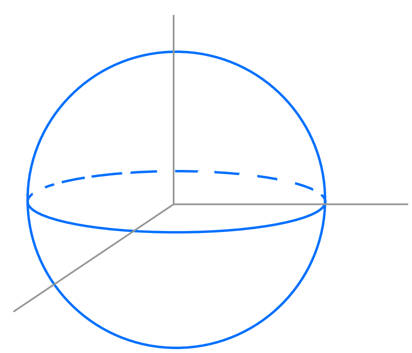
$B = \{\underset{\sim}{x} \in \mathbb{R}^3 : x_3 = 0\}$



$C = \{\underset{\sim}{x} \in \mathbb{R}^2 : \underbrace{x_1^2 + x_2^2}_{\|\underset{\sim}{x}\|^2} = 1\}$



$D = \{\underset{\sim}{x} \in \mathbb{R}^3 : \underbrace{x_1^2 + x_2^2 + x_3^2}_{\|\underset{\sim}{x}\|^2} = 1\}$

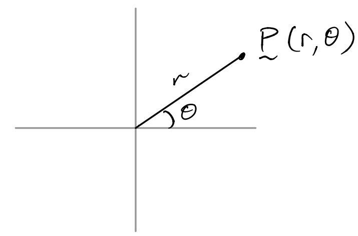


Coordinate systems

Sometimes it is convenient to represent a point/vector $\underset{\sim}{P}$ in $\mathbb{R}^2$ in polar coordinates (r, θ) (instead of cartesian (x,y))

r = distance from $\underset{\sim}{O}$ $\quad r \geq 0$

θ = angle with x-axis $\quad \theta \in [0, 2\pi)$



basic trigonometry gives the coordinate transformation

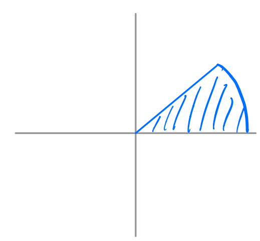
$$x = r\cos\theta$$
$$y = r\sin\theta$$

For example, this gives another way to describe the set C from above

$$C = \{ \underset{\sim}{P}(r,\theta) \in \mathbb{R}^2 : r = 1 \}$$

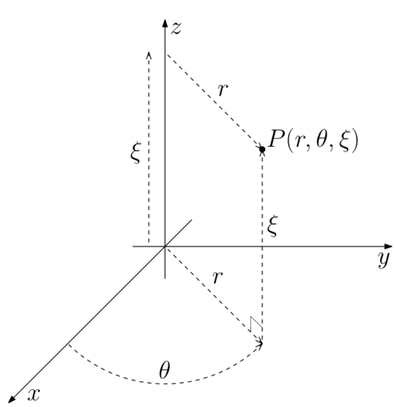
and makes it easy to describe sectors:

$$S_1 = \{ \underset{\sim}{P}(r,\theta) \in \mathbb{R}^2 : r \leq 1,\ \theta \in [0, \tfrac{\pi}{4}] \}$$



In $\mathbb{R}^3$ there are <u>two</u> alternative coordinate systems that are frequently used:

cylindrical coordinates (r, θ, ξ) $\quad$ r, theta, xi



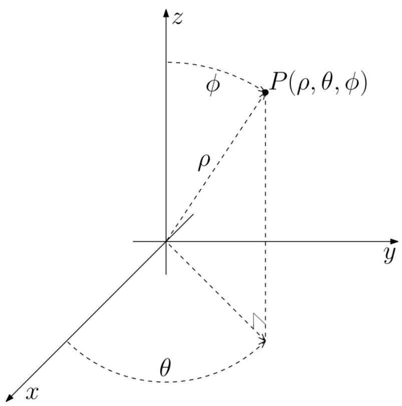
$\left.\begin{array}{l} r = \text{distance from } z \text{ axis} \\ \theta = \text{angle with } x \text{ axis in } xy\text{-plane} \end{array}\right\}$ polar

ξ = height above xy plane $(=z)$

$r \geq 0, \quad \theta \in [0, 2\pi), \quad \xi \in \mathbb{R}$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\xi = z$$

Spherical coordinates (ρ, θ, ϕ) rho, theta, phi



$\rho =$ distance from $\underset{\sim}{O}$

$\theta =$ xy-plane angle with x-axis

$\phi =$ drop down angle from z-axis

$\rho \geq 0, \quad \theta \in [0, 2\pi), \quad \phi \in [0, \pi)$

$$x = \rho \cos\theta \sin\phi$$

$$y = \rho \sin\theta \cos\phi$$

$$z = \rho \cos\phi$$

A function between two sets X, Y is an assignment of one element of Y to each element of X

Notation: $f : X \to Y$
$a \mapsto f(a)$ $\qquad a \in X$ maps to $f(a) \in Y$

$f(a)$ is called the image of a

X is called the domain of f

the range of f is the set of all images $\{ f(a) \in Y : a \in X \}$

Example: $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ $\quad$ domain$(f) = \mathbb{R}$
$x \mapsto x^2$ $\quad$ range$(f) = \{ x \in \mathbb{R} : x \geq 0 \}$

Let $D \subset \mathbb{R}$, a vector valued function of one variable is a function $\underset{\sim}{r} : D \longrightarrow \mathbb{R}^n$

$$t \longmapsto \underset{\sim}{r}(t) = (r_1(t), r_2(t), \dots, r_n(t))$$

coordinate functions of $\underset{\sim}{r}$

typical application: t represents time
$\underset{\sim}{r}(t)$ position of an object at time t

Examples

$\underset{\sim}{r} : [0,1] \to \mathbb{R}^2$
$t \mapsto (1+t, t)$

$c : [0, 2\pi] \longrightarrow \mathbb{R}^2$, $\quad c(t) = (\cos t, \sin t)$

$\gamma : [0, 4\pi] \longrightarrow \mathbb{R}^3$
$t \longmapsto (\cos t, \sin t, t)$

we can visualise such functions using parametric plots:
plot the point $\underset{\sim}{r}(t) \in \mathbb{R}^n$ for each value of t

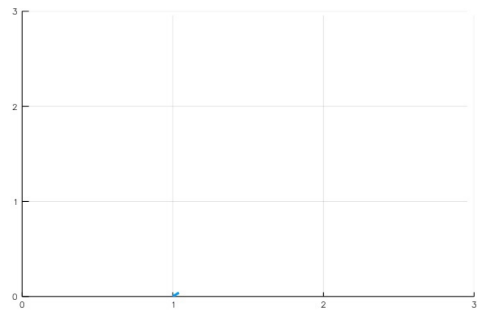
eg:

$$\underset{\sim}{r} : [0,1] \to \mathbb{R}^2$$

$$\underset{\sim}{r}(t) = (1+t, t)$$

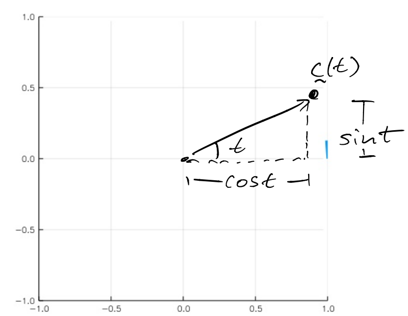
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

starting point   direction vector



$$c : [0, 2\pi] \to \mathbb{R}^2$$

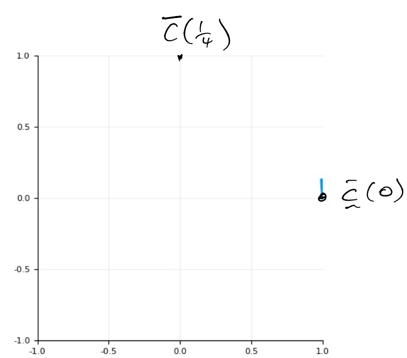
$$c(t) = (\cos t, \sin t)$$



$$\bar{c} : [0,1] \to \mathbb{R}^2$$

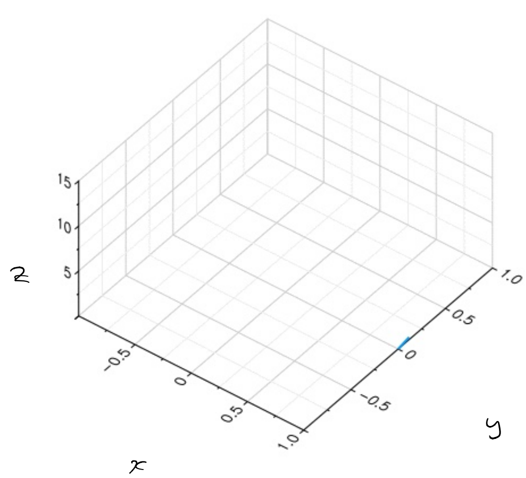
$$\theta \mapsto (\cos(2\pi\theta), \sin(2\pi\theta))$$

same curve, different parametrization.



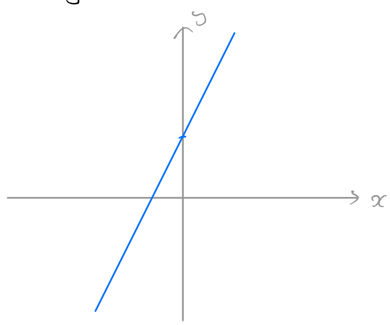
$$\gamma(t) = (\cos t, \sin t, t)$$

helix!



comparison with graphs: consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$

usually visualise f using its graph:



i.e. the set of points $(x, y) \in \mathbb{R}^2$ with

$y = 2x + 1$

or

$$\text{graph } f = \{ (x, f(x)) : x \in \mathbb{R} \} \subset \mathbb{R}^2$$

writing t instead of x: graph $f = \{ (t, f(t)) : t \in \mathbb{R} \}$

and this is just the parametric plot of the function $\underset{\sim}{r}$ defined by

$$\begin{aligned} \underset{\sim}{r} : \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto (t, f(t)) \end{aligned}$$

A real-valued function of two variables is a map $D \to \mathbb{R}$

where $D \subset \mathbb{R}^2$, so $f: D \to \mathbb{R}$

$$(x,y) \mapsto f(x,y) \qquad (\text{not } f((x,y)))$$

eg: $f(x,y) = x^2 + y^2$

$g(x,y) = e^x + x \sin y$

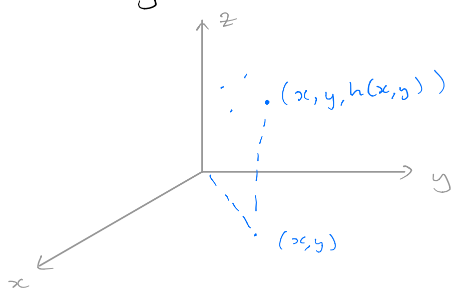
example application: (x,y) : position on some terrain

$h(x,y)$: height above sea level at (x,y)

usually visualise using graphs:

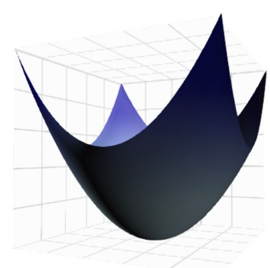
$$\text{graph } h = \{ (x, y, h(x,y)) : (x,y) \in D \} \subset \mathbb{R}^3$$

to each (x,y) we assign $z = h(x,y)$

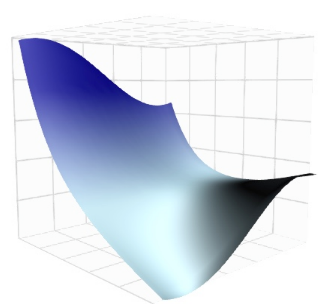


surface plots!

eg: $f(x,y) = x^2 + y^2$



$g(x,y) = e^x + x \sin y$



alternative: contour plots - 2D visualisation

eg: topographic maps

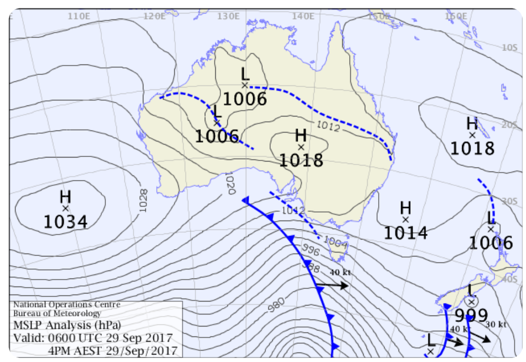
$h(x,y)$ height above sea level of the land

contours are curves of equal elevation

weather maps

$f(x,y)$ = mean air pressure

contours (isobars) are curves of equal pressure.



curves of equal * are called level curves

i.e. for $f : D \to \mathbb{R}$ the level curve at level $k \in \mathbb{R}$ is

$$\{ (x,y) \in D : f(x,y) = k \}$$

(if k is not in range f then this is the empty set)

Example

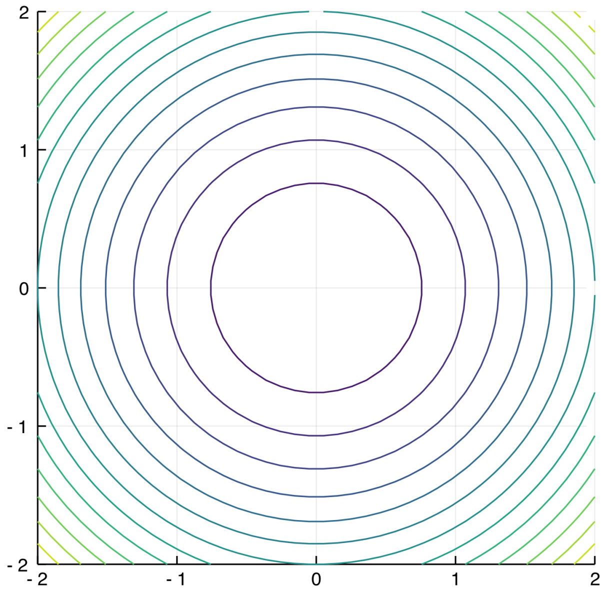
$f(x,y) = x^2 + y^2$

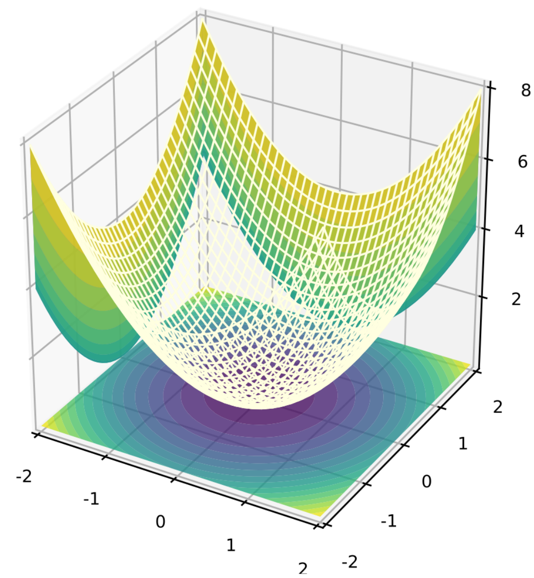
$k=1$ level curve : $f(x,y) = 1 = x^2+y^2$

$x^2+y^2 = 1$ ← circle of radius 1

$k=2$ level curve : $f(x,y) = 2 = x^2+y^2$ circle of radius $\sqrt{2}$

...





functions of three or more variables

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x_1, x_2, x_3) \longmapsto f(x_1, x_2, x_3)$$

graph $f = \{(x_1, x_2, x_3, f(x_1, x_2, x_3))\} \subset \underline{\mathbb{R}^4}$

not possible to plot graph f

but we can plot level surfaces $f(x_1, x_2, x_3) = k$

eg: $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

level surface at $k = 1$:

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : f(\underset{\sim}{x}) = x_1^2 + x_2^2 + x_3^2 = 1\}$$

- this is a sphere!

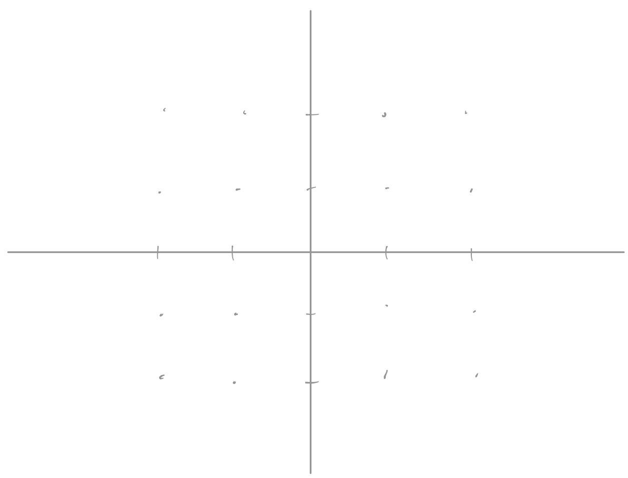
A map $\underset{\sim}{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called a vector field on $\mathbb{R}^2$

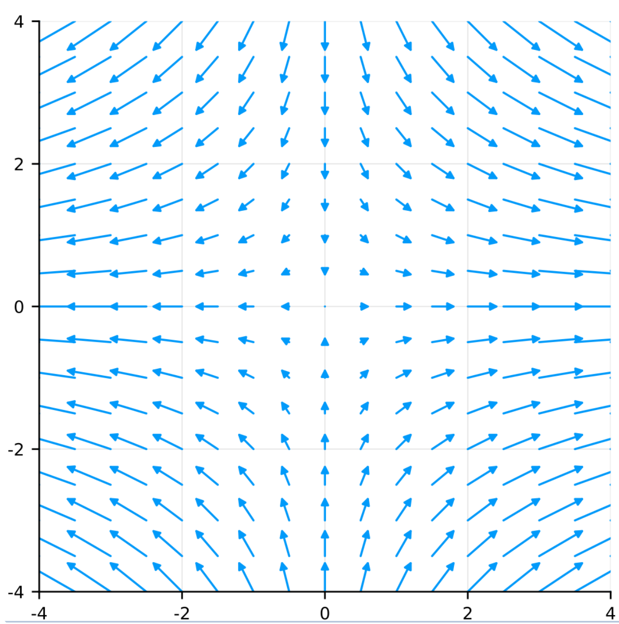
$\underset{\sim}{f}$ assigns a vector in $\mathbb{R}^2$ to each $(x,y) \in \mathbb{R}^2$

$$\underset{\sim}{f}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

vector fields are usually visualised by drawing a scaled arrow representing the vector $\underset{\sim}{f}(x,y)$ with its tail at the point (x,y), for a finite number of (x,y).

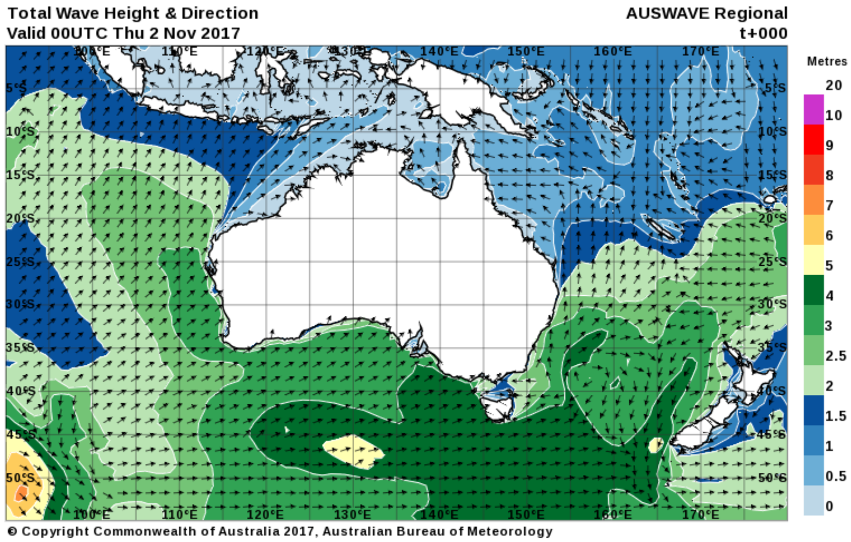
Example $\underset{\sim}{f}(x,y) = (2x, -y)$



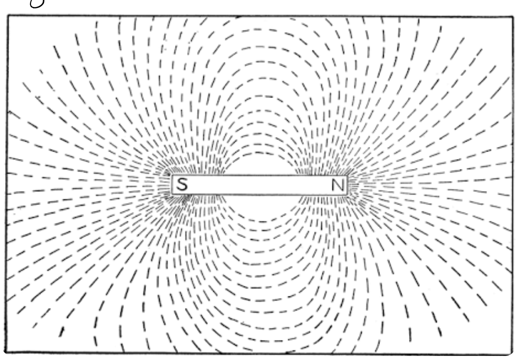


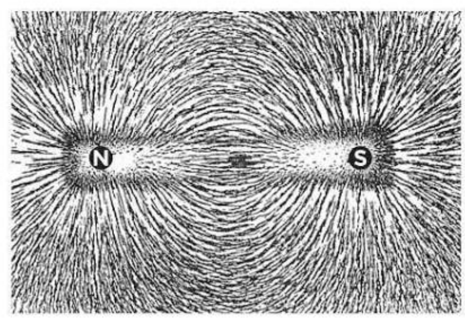
scalefactor $= 0.1$.

Examples   Swell charts



Magnetic fields



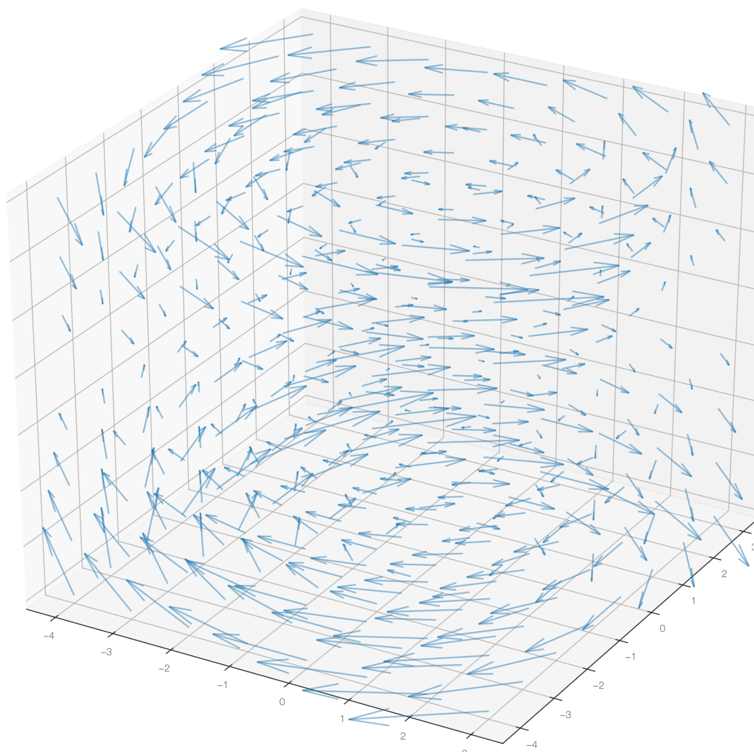


Vector fields in 3D

$$\underset{\sim}{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\underset{\sim}{f}(x, y, z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

Example $\quad \underset{\sim}{f}(x,y,z) = (-yz, \; xz, \; 0)$



Applications :

fluid flow $\quad \underset{\sim}{f}(x,y,z)$ is the velocity of a particle at (x,y,z)

force fields $\quad$ (electric, magnetic, gravitational)

$\underset{\sim}{f}(x,y,z)$ is the strength & direction of a force at (x,y,z).