MATH 1011 : MULTIVARIABLE CALCULUS

INTRODUCTION : what is multivariable calculus?

Calculus

limits, continuity, differentiation and integration of functions

Variables

$\mathbb{R}$ real numbers

$\mathbb{R}^2$ = set of pairs x, y such that x, y are elements of $\mathbb{R}$

= $\{ \qquad \}$ notation:

$\mathbb{R}^3$ = $\{ \qquad \}$ $\underset{\sim}{x}$ =

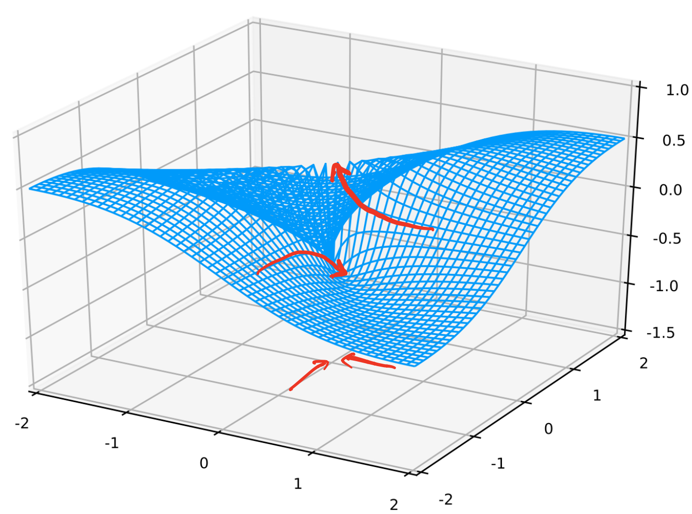
$\vdots$ triples

$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) : x_1, x_2 \dots x_n \in \mathbb{R} \}$

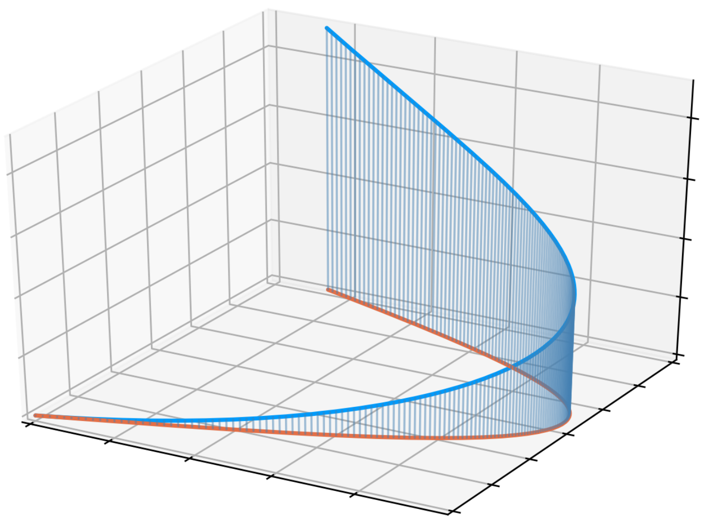
Multivariable calculus

extend limits, continuity, differentiation and integration to functions

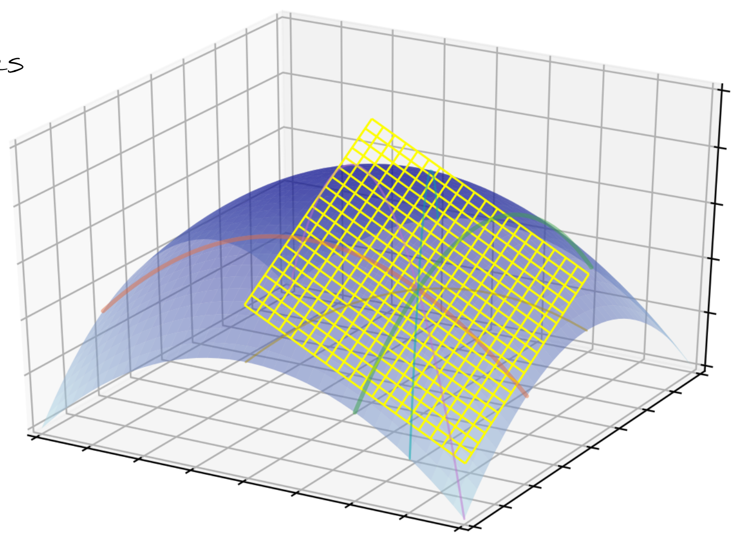
eg: limits of functions of two variables



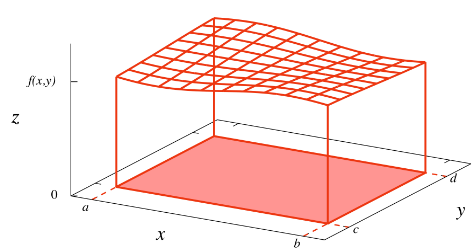
line integrals



tangent planes



double integrals



and other things too...

$$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_1, x_2 \ldots x_n \in \mathbb{R} \}$$

n-tuples

$\mathbb{R}^n$ is a vector space:

zero vector

addition $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^n$ $\underset{\sim}{u} = (u_1, u_2, \ldots, u_n)$ $\underset{\sim}{v} = (v_1, v_2, \ldots, v_n)$

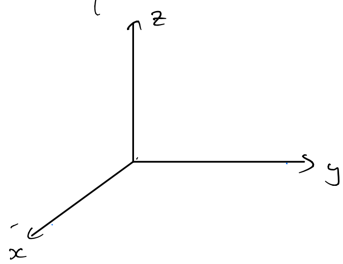
scalar multiplication $\alpha \in \mathbb{R}$

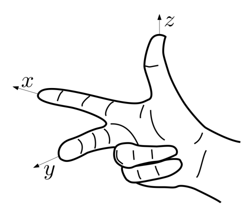
Applications

an element $\underset{\approx}{x} \in \mathbb{R}^n$ might represent:

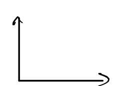
- coordinates of a point P in space

eg: $(1,1,1)$

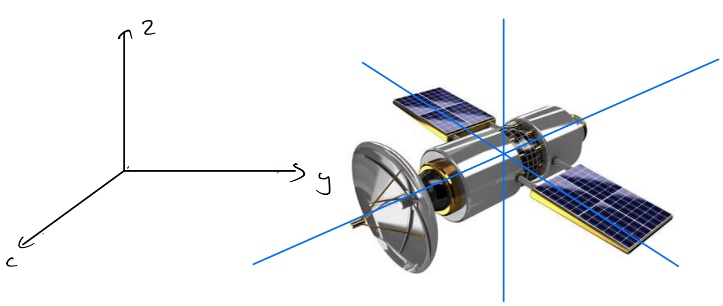




- or a vector (instructions for getting to a point)
- coordinates of a point in spacetime



- location and orientation of a satellite

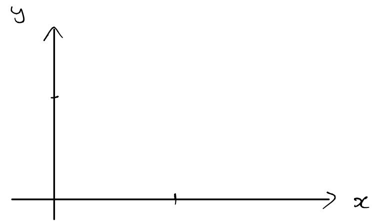


Note: since $\mathbb{R}^n$ is a vector space, its elements are called vectors, even if they are being used to represent points.

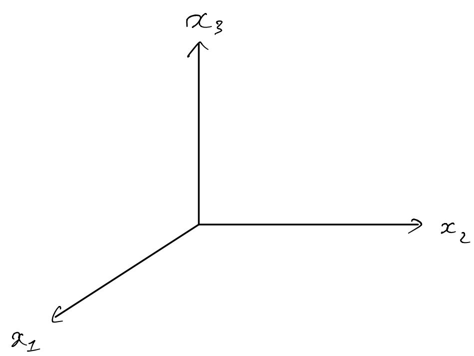
subsets of $\mathbb{R}^n$ (examples)

$$A = \{ (x,y) \in \mathbb{R}^2 : \qquad \text{and} \qquad \}$$

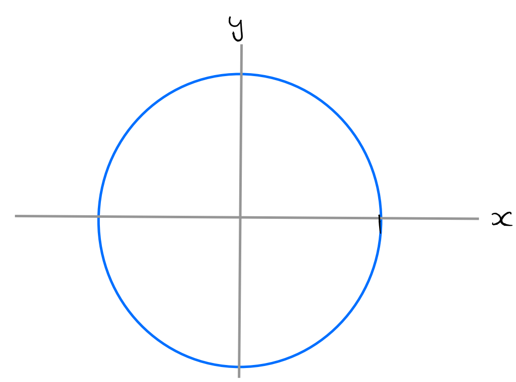
graphically



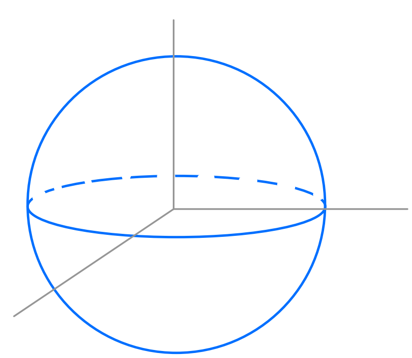
$$B = \{ \underset{\sim}{x} \in \mathbb{R}^3 : x_3 = 0 \}$$



$$C = \{ \underset{\sim}{x} \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$$



$$D = \{ \underset{\sim}{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \}$$

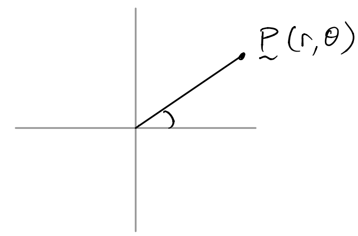


Coordinate systems

Sometimes it is convenient to represent a point/vector $\underset{\sim}{P}$ in $\mathbb{R}^2$ in polar coordinates (instead of cartesian)

r = distance from $\underset{\sim}{O}$

θ = angle with x-axis



basic trigonometry gives the coordinate transformation

$x =$

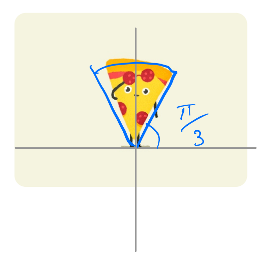
$y =$

For example, this gives another way to describe the set C from above

$$C = \{ \qquad \qquad \qquad \}$$

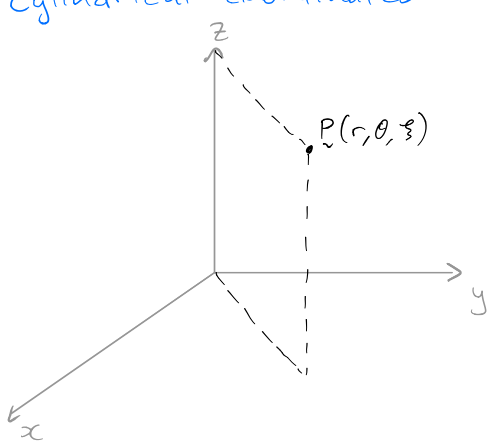
and makes it easy to describe sectors:

$$S_1 = \{ \underset{\sim}{P}(r,\theta) \in \mathbb{R}^2 \ : \qquad \qquad \qquad \}$$



In $\mathbb{R}^3$ there are two alternative coordinate systems that are frequently used:

cylindrical coordinates (r, θ, ξ) r, theta, xi



r = distance from z axis

θ = angle with x axis in xy-plane

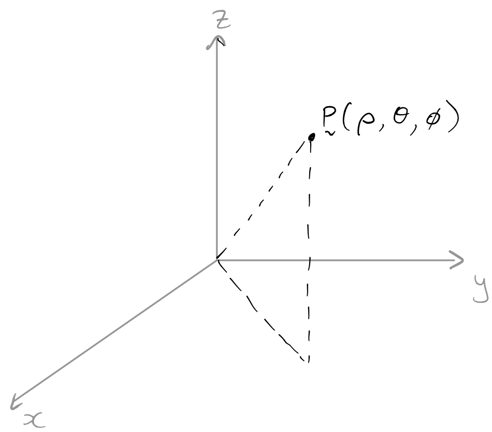
ξ = height above xy plane

$x =$

$y =$

$\xi =$

Spherical coordinates (ρ, θ, ϕ) rho, theta, phi



ρ = distance from $\underset{\sim}{O}$

θ = xy-plane angle with x-axis

ϕ = drop down angle from z-axis

$\rho \geq$, $\theta \in$, $\phi \in$

$x =$

$y =$

$z =$

A real-valued function of two variables is a map $D \to \mathbb{R}$

where $D \subset \mathbb{R}^2$ so $f: D \to \mathbb{R}$

eg: $f(x,y)$

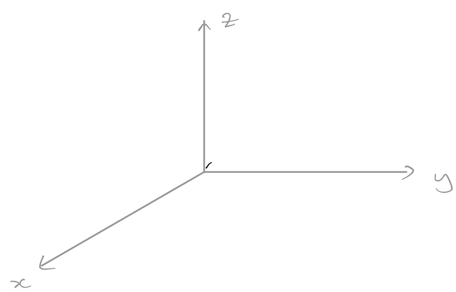
$g(x,y)$

example application: (x,y) : position on the earth's surface

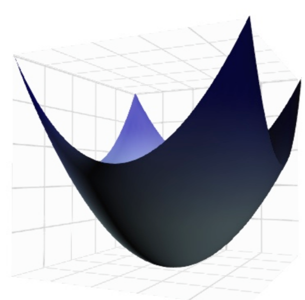
$h(x,y)$: height above sea level at that point

usually visualise using graphs:

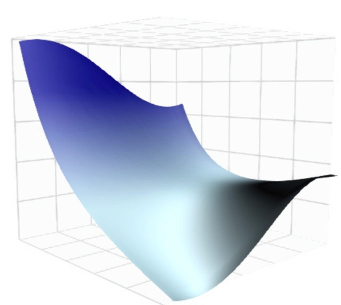
$$\text{graph } h = \{ (x, y, h(x,y)) : (x,y) \in D \} \subset \mathbb{R}^3$$



eg: $f(x,y) = x^2 + y^2$



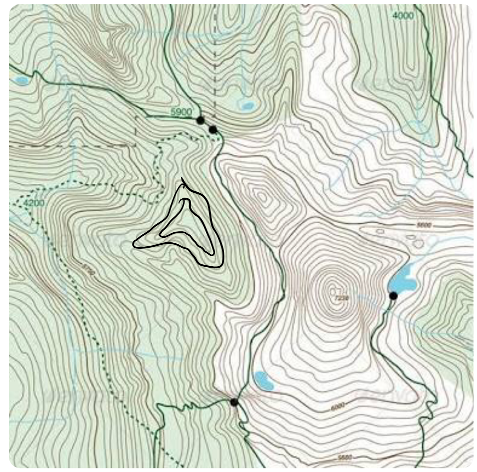
$g(x,y) = e^x + x \sin y$



alternative: contour plots - 2D visualisation

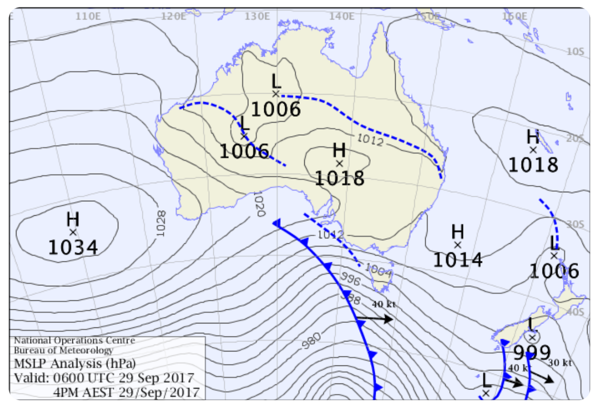
eg: topographic map

$h(x,y)$



weather maps

$p(x,y)$



curves of equal * are called level curves

i.e. for $f: D \to \mathbb{R}$ the level curve at level $k \in \mathbb{R}$ is

(if k is not in range f then this is the empty set)

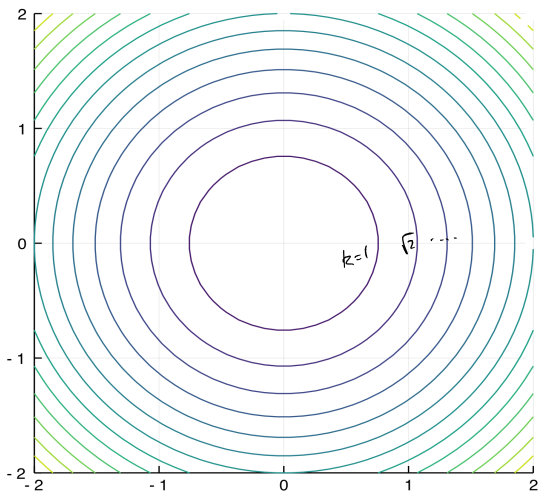
Example

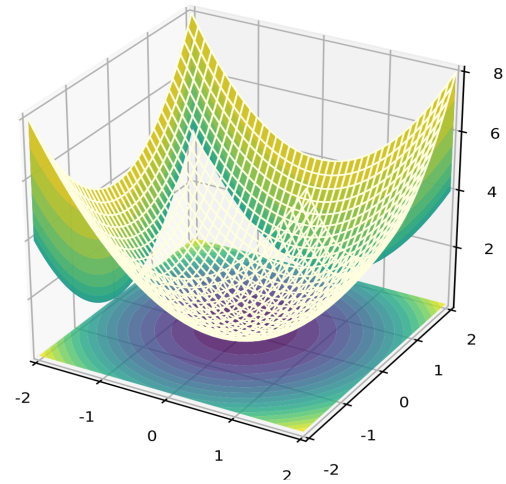
$f(x,y) = x^2 + y^2$

$k=1$ level curve : circle of radius 1

$k=2$ level curve : circle of radius $\sqrt{2}$

$k \ldots$





functions of three or more variables

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x_1, x_2, x_3) \mapsto f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$\text{graph } f = \{(x_1, x_2, x_3, f(x_1, x_2, x_3)) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

- not possible to visualise graph f

  but we can plot level surfaces

eg: $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

level surface at $k=1$:

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : f(\underset{\sim}{x}) = x_1^2 + x_2^2 + x_3^2 = 1\}$$

A function between two sets X, Y is an assignment of one element of Y to each element of X

Notation:

is called the image of a

is called the domain of f

the range of f is the set of all images

Example:

Let $D \subset \mathbb{R}$, a vector valued function of one variable is a function

$$\underset{\sim}{r} : D \longrightarrow \mathbb{R}^n$$
$$t \longmapsto \underset{\sim}{r}(t) = (r_1(t), r_2(t), \dots, r_n(t))$$

typical application: t represents time

$\underset{\sim}{r}(t)$ position of an object at time t

Examples

$\underset{\sim}{r} : [0,1] \to \mathbb{R}^2$

$c : [0, 2\pi] \longrightarrow \mathbb{R}^2$

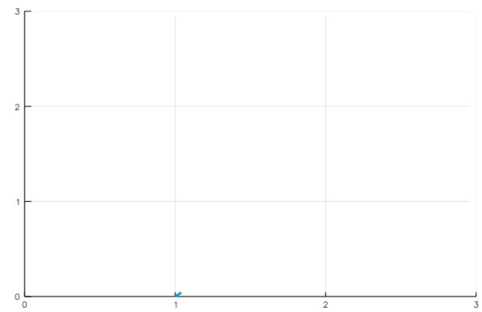
$\gamma : [0, 4\pi] \longrightarrow \mathbb{R}^3$

we can visualise such functions using parametric plots:

plot the point $\underset{\sim}{r}(t) \in \mathbb{R}^n$ for each value of t

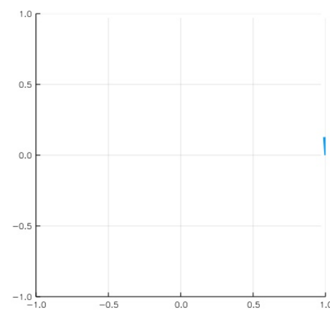
eg: $\underset{\sim}{r} : [0,1] \to \mathbb{R}^2$

$\underset{\sim}{r}(t) = (1+t, t)$



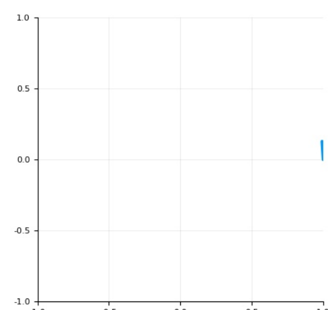
$c : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$c(t) = (\cos t, \sin t)$

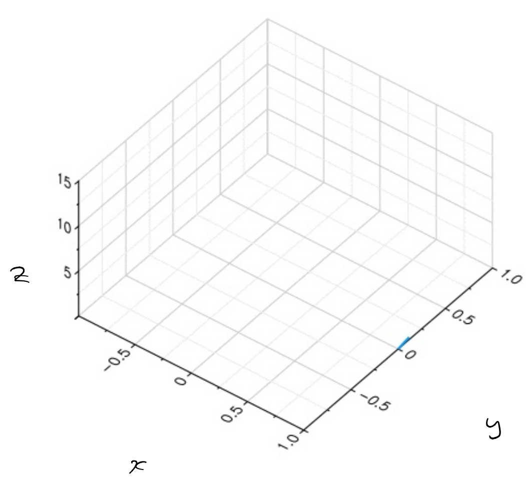


$\bar{c} : [0, 1] \to \mathbb{R}^2$

$\theta \longmapsto (\cos(2\pi\theta), \sin(2\pi\theta))$

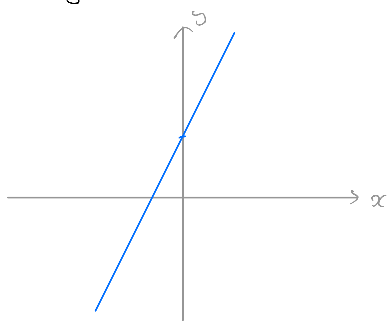


$\gamma(t) = (\cos t, \sin t, t)$



comparison with graphs: consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$

usually visualise f using its graph:



i.e. the set of points with

or

graph f =

writing t instead of x: graph f =

and this is just the parametric plot of the function $\underset{\sim}{r}$ defined by

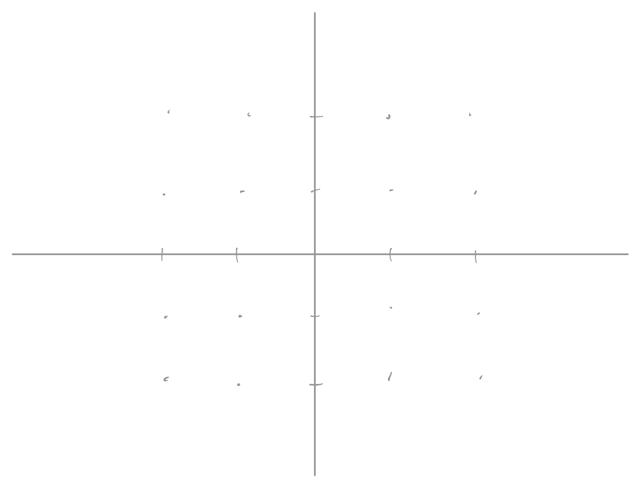
$$\underset{\sim}{r} : \mathbb{R} \to \mathbb{R}^2$$

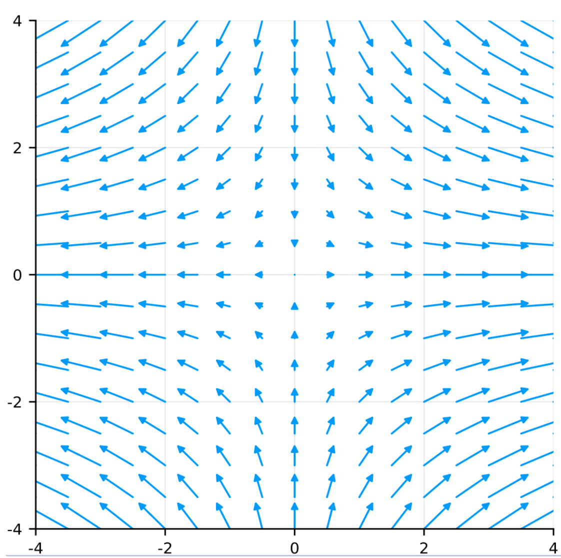
A map is called a vector field on $\mathbb{R}^2$

$\underset{\sim}{f}$ assigns a vector in $\mathbb{R}^2$ to each

Vector fields are usually visualised by drawing a scaled arrow representing the vector $\underset{\sim}{f}(x,y)$ with its tail at the point (x,y), for a finite number of (x,y).

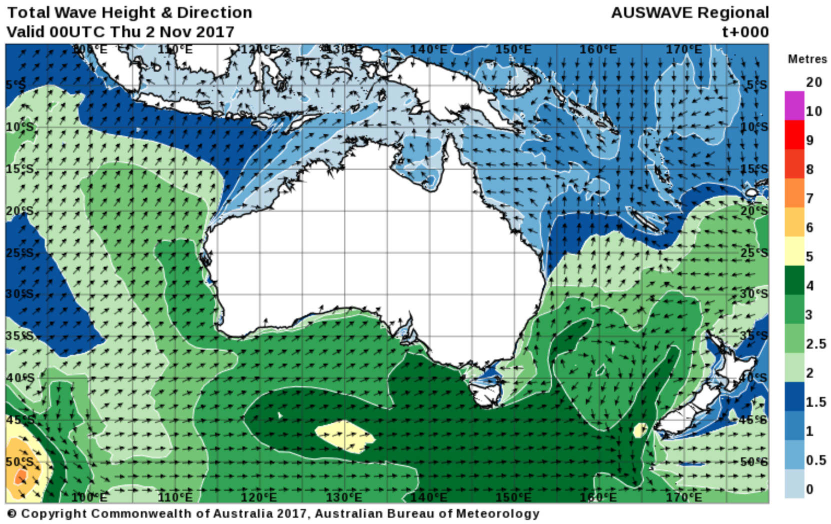
Example $\underset{\sim}{f}(x,y) = (2x,-y)$



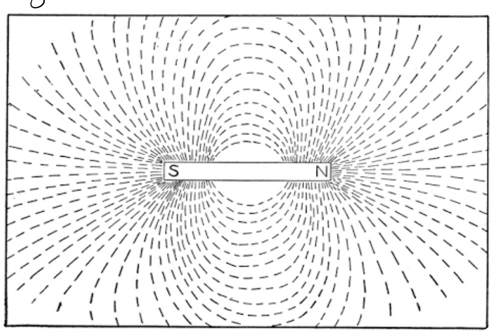


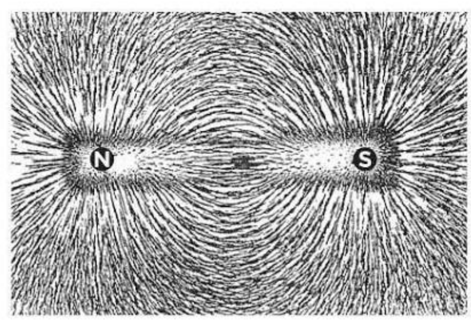
scalefactor $= 0.1$.

Examples Swell charts



Magnetic fields



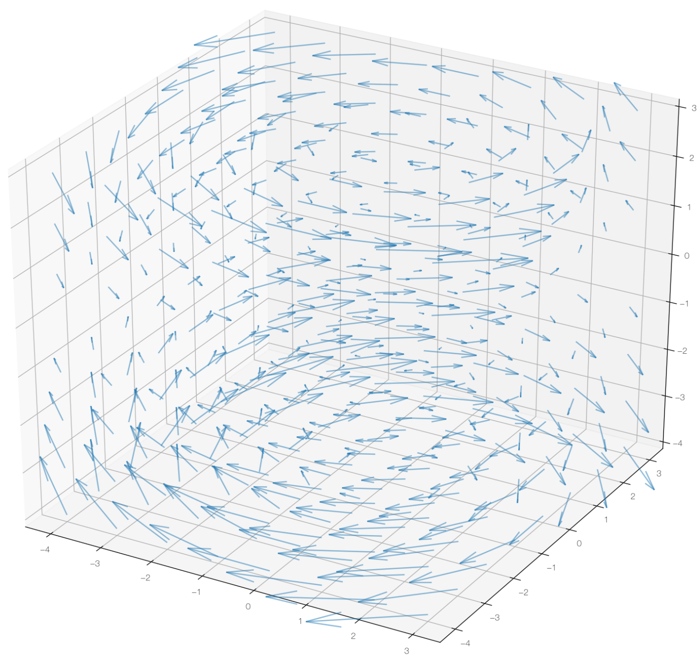


Vector fields in 3D

$$\underset{\sim}{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\underset{\sim}{f}(x, y, z) =$$

Example $\quad \underset{\sim}{f}(x, y, z) =$



Applications :

fluid flow $\quad \underset{\sim}{f}(x, y, z)$

force fields $\quad$ (electric, magnetic, gravitational)

$\underset{\sim}{f}(x, y, z)$