The limit at a point (intuitive definition)

$f : (a,b) \to \mathbb{R}$, $c \in (a,b)$

we say the limit as x approaches c of $f(x)$ is L

notation: $\lim_{x \to c} f(x) = L$

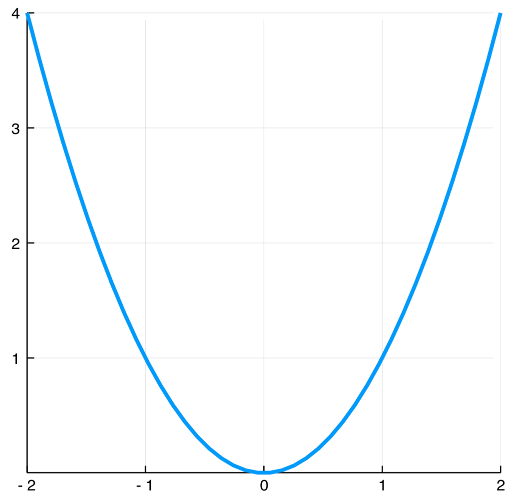
if when x is close to c $f(x)$ is close to L, and we can make $f(x)$ as close as we like to L by taking x sufficiently close to c.

Examples

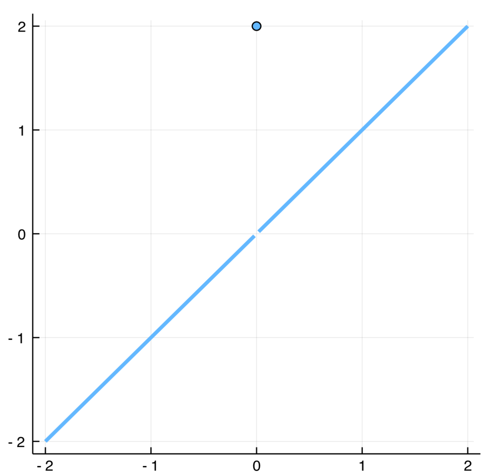
$f(x) = x^2$

$\lim_{x \to 0} f(x) = 0$

$\lim_{x \to \sqrt{2}} f(x) = 2$



$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} 2 & \text{if } x = 0 \\ x & \text{otherwise} \end{cases}$ (piecewise function)

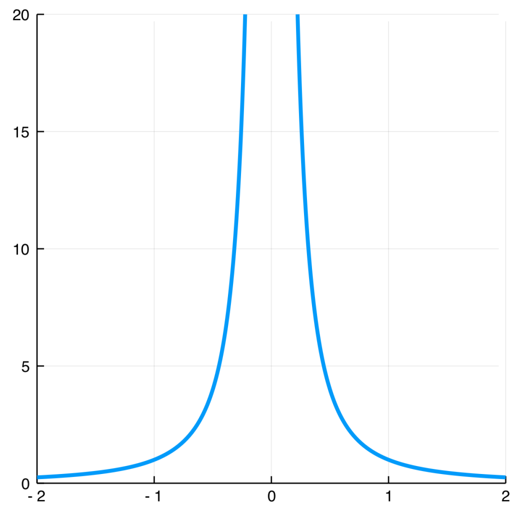


$\lim_{x \to 0} f(x) = 0$

$(\neq 2 \, !)$

$$f(x) = \frac{1}{x^2}$$

$$\lim_{x \to 0} f(x) =$$ 



this is an example of a function diverging to infinity:

we say f diverges to infinity at c if we can make $f(x)$ as large as we like by taking x sufficiently close to c

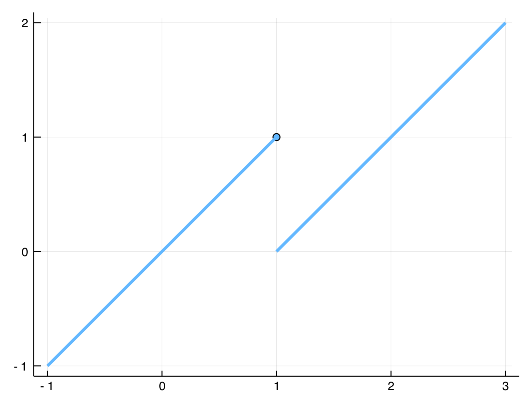
notation: $\lim_{x \to c} f(x) = \infty$

remember though: ∞ is NaN !

$$f(x) = \begin{cases} x & \text{when } x \leq 1 \\ x - 1 & x > 1 \end{cases}$$

$$\lim_{x \to 1} f(x) = ?$$

doesn't exist



one sided limits:

$x \to 1$ with $x < 1$

$$\lim_{x \to 1^-} f(x) = 1$$

$x \to 1$ with $x > 1$

$$\lim_{x \to 1^+} f(x) = 0$$

$\lim_{x \to c} f(x)$ exists $\Leftrightarrow$ both one-sided limits exist and are equal.

the limit as $x \to \infty$

we write $\lim\limits_{x \to \infty} f(x) = L$ if we can make $f(x)$ as close to L as we like by taking x sufficiently large.

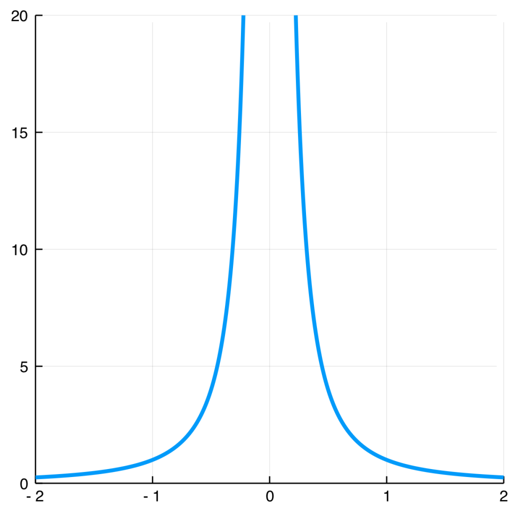
$\lim\limits_{x \to -\infty} f(x) = L$ similar.

Examples

$$f(x) = \frac{1}{x^2}$$

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x^2} = 0$$

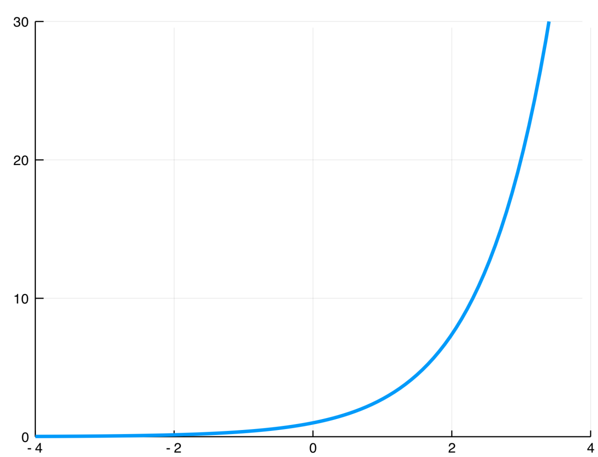


in fact: $\lim\limits_{x \to \infty} \frac{1}{x^p} = 0$ for any constant $p > 0$

$$f(x) = e^x$$

$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to -\infty} e^x = 0$$



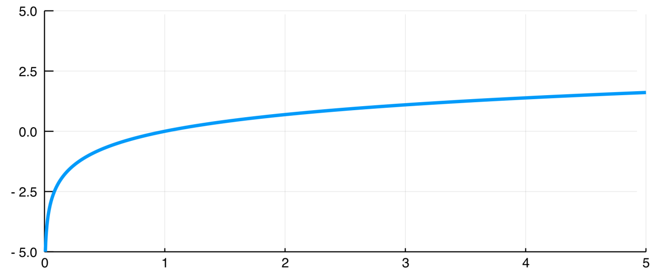
$$f(x) = \ln x$$

$$\lim_{x \to 0^+} \ln x = -\infty$$

$$\lim_{x \to \infty} \ln x = \infty$$

not obvious!

proof later.



Limit laws

If f and g are functions $(b,c) \to \mathbb{R}$, $a \in (b,c)$ and the limits $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ both exist, then

- $$\lim_{x \to a} \left(f(x) + g(x)\right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

- for any constant $k \in \mathbb{R}$
$$\lim_{x \to a} k\, f(x) = k \lim_{x \to a} f(x)$$

- $$\lim_{x \to a} \left(f(x) \cdot g(x)\right) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

- if $\lim_{x \to a} g(x) \neq 0$, then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

These laws also hold for one-sided limits $\lim_{x \to a^+}$, $\lim_{x \to a^-}$ and limits at infinity $\lim_{x \to \pm\infty}$

Examples

1. $$\lim_{x \to 2} (x^2 + 2x) = \lim_{x \to 2} x^2 + 2 \lim_{x \to 2} x = 4 + 4 = 8$$

2. $$\lim_{x \to \pi} \frac{x^2}{\cos x}$$

$\lim_{x \to \pi} x^2 = \pi^2$, $\lim_{x \to \pi} \cos x = 1$

both exist, nonzero

therefore $$\lim_{x \to \pi} \frac{x^2}{\cos x} = \frac{\pi^2}{1} = \pi^2$$

3. $\lim_{x \to -2} \dfrac{x^2+2x}{x+2} \neq \dfrac{\lim_{x\to -2} x^2+2x}{\lim_{x \to -2} x+2}$ ← this limit is zero

instead: $\dfrac{x^2+2x}{x+2} = \dfrac{x\cancel{(x+2)}}{\cancel{x+2}} = x$

$$\lim_{x\to -2} \frac{x^2+2x}{x+2} = \lim_{x\to -2} x = -2.$$

4. $\lim_{x\to\infty} 1-2x = -\infty$ $\qquad \lim_{x\to\infty} 1+2x = \infty$

$$\lim_{x\to\infty} (1-2x + 1+2x) = \lim_{x\to\infty} 2 = 2$$

$$\neq \lim_{x\to\infty} 1-2x + \lim_{x\to\infty} 1+2x$$

$$(-\infty + \infty)$$

(don't treat ∞ as a number)

5. $\lim_{x\to\infty} \dfrac{x^2+2x+1}{2x^2+x+7} = \lim_{x\to\infty} \dfrac{\cancel{x^2}(1+2/x+1/x^2)}{\cancel{x^2}(2+1/x+7/x^2)}$

numerator:

$$\lim_{x\to\infty}\left(1+\frac{2}{x}+\frac{1}{x^2}\right) = \lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{2}{x} + \lim_{x\to\infty}\frac{1}{x^2}$$

$$= \underline{1} + 0 + 0$$

denominator:

$$\lim_{x\to\infty}(2+1/x+7/x^2) = \lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{7}{x^2}$$

$$= 2 + 0 + 0$$

denominator $\neq 0$:

$$\lim_{x\to\infty}\frac{x^2+2x+1}{2x^2+x+7} = \frac{1}{2}$$

The squeeze theorem

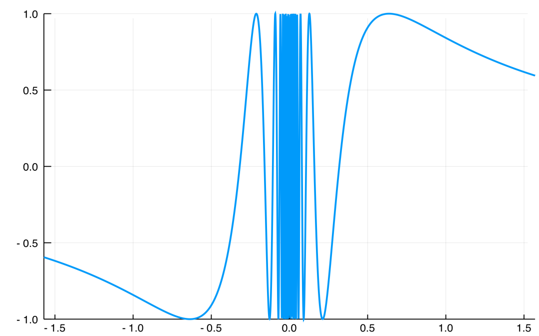
Let $\underline{f,g,h} : (a,b) \to \mathbb{R}$ and suppose

$$f(x) \le g(x) \le h(x) \qquad \text{for all } x \in (a,b)$$

if $c \in (a,b)$ and $f(x) \xrightarrow[x\to c]{} L$, $g(x) \to L$, $h(x) \xrightarrow[x\to c]{} L$

then $g(x) \xrightarrow[x\to c]{} L$ also.

Example $\lim\limits_{x\to 0} \underline{x^2}\ \underline{\sin\left(\frac{1}{x}\right)}$



$\lim\limits_{x\to 0} x^2 = 0$

$\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right) =$ doesn't exist

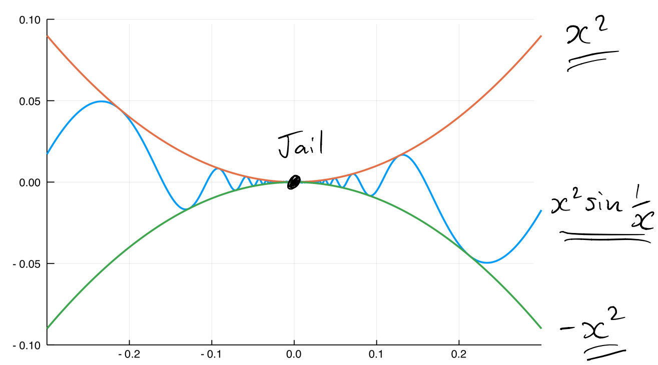
but x^2 dominates:

$$-1 \le \sin\frac{1}{x} \le 1$$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

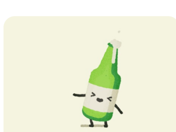
$$\underset{x\to 0}{\searrow} \quad \downarrow \quad \underset{x\to 0}{\swarrow}$$

$$0$$



so $\lim\limits_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the squeeze theorem

– a.k.a. "two policemen and a drunk" theorem



$x^2 \sin \frac{1}{x}$

$x^2, \ -x^2$

The theorem also holds for limits at infinity $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$

Example $\lim_{x\to\infty} \frac{2\cos x}{x}$

$$-1 \le \cos x \le 1$$

$$-2 \le 2\cos x \le 2$$

$x>0$: $$-\frac{2}{x} \le \frac{2\cos x}{x} \le \frac{2}{x}$$

$$-\frac{2}{x} \xrightarrow{x\to\infty} 0, \qquad \frac{2\cos x}{x} \to 0, \qquad \frac{2}{x} \xrightarrow{x\to\infty} 0$$

$\lim_{x\to\infty} \frac{2\cos x}{x} = 0$ by the squeeze theorem.

First recall the intuitive definition:

$f: (a,b) \to \mathbb{R}$, $c \in (a,b)$ $\quad \lim_{x \to c} f(x) = L$

if when x is close to c $f(x)$ is close to L,
and we can make $f(x)$ as close as we like to L by taking x sufficiently close to c.

what exactly is meant by "close" and "as close as we like"?

The precise definition of a limit at a point:

$\lim_{x \to c} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

if $|x-c| < \delta$ then $|f(x) - L| < \varepsilon$

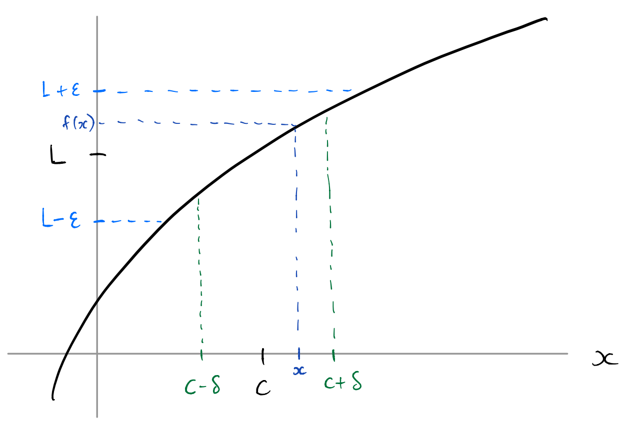
i.e. if $-\delta < x - c < \delta$ $\qquad$ $-\varepsilon < f(x) - L < \varepsilon$

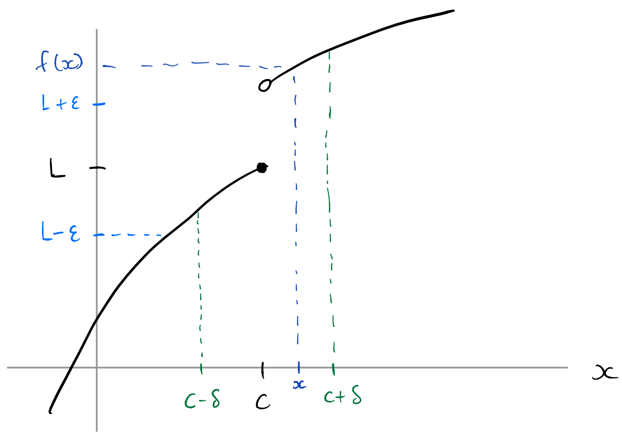
$c - \delta < x < c + \delta$ then $L - \varepsilon < f(x) < L + \varepsilon$

for ε, δ small : "x is close to c" $\qquad$ "$f(x)$ is close to L"

Since it is for all ε, ε can be as small as we like and a δ still exists — this captures the "as close as we like" part.

graphically





Example $\quad f(x) = 2x$

expect $\lim_{x \to 1} f(x) = 2$ – let's prove it

need to show: for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

if $|x-1| < \delta$ then $|f(x) - 2| < \varepsilon$

Since we need to prove that for all ε there exists δ,

we must find an expression for δ in terms of ε.

As a first guess: let $\delta = \varepsilon$

if $|x-1| < \delta = \varepsilon$ then $-\varepsilon < x - 1 < \varepsilon$

$$\Rightarrow \quad -2\varepsilon < 2x - 2 < 2\varepsilon$$

$$\Rightarrow \quad -2\varepsilon < f(x) - 2 < 2\varepsilon$$

$$\Rightarrow \quad |f(x) - 2| < 2\varepsilon$$

this is close, but not quite what is needed ($|f(x) - 2| < \varepsilon$)

Can fix it by letting $\delta = \varepsilon/2$ instead:

if $|x-1| < \delta = \frac{\varepsilon}{2}$ then $-\frac{\varepsilon}{2} < x - 1 < \varepsilon/2$

$$\Rightarrow \quad -\varepsilon < 2x - 2 < \varepsilon$$

$$\Rightarrow \quad -\varepsilon < f(x) - 2 < \varepsilon$$

$$\Rightarrow \quad |f(x) - 2| < \varepsilon$$

therefore, for all $\varepsilon > 0$ there exists δ (for example $\delta = \frac{\varepsilon}{2}$)
such that $|x-1| < \delta \Rightarrow |f(x)-2| < \varepsilon$

i.e. $\lim_{x \to 1} f(x) = 2$. $\square$

Definition of the limit at infinity

Given $f : (c, \infty) \to \mathbb{R}$ write $\lim_{x \to \infty} f(x) = L$ if for all $\varepsilon > 0$ there exists b such that if $x \geq b$ then $|f(x) - L| < \varepsilon$

Definition of diverging to infinity

$\lim_{x \to a^-} f(x) = \infty$ if for any $M > 0$ there exists $\delta > 0$ such that if $x < a$ and $|x-a| < \delta$ then $f(x) > M$.

Diverging to infinity at infinity

$\lim_{x \to \infty} f(x) = \infty$ if for any $M > 0$ there exists b such that if $x \geq b$ then $f(x) > M$.

Example $\lim_{x \to \infty} \ln x = \infty$

proof: given $M > 0$, let $b = e^M$

Note that: $\frac{d}{dx} \ln x = \frac{1}{x} > 0$ for all $x > 0$ so $\ln x$ is an increasing function. Therefore taking logs preserves the inequality:

$$x \geq b \Rightarrow \ln x > \ln b = \ln e^M = M$$

$$\ln x > M. \quad \text{so} \quad \lim_{x \to \infty} \ln x = \infty$$

limits of vector valued functions

$$\underset{\sim}{r}(t) = (r_1(t), r_2(t), r_3(t))$$

$$\lim_{t \to c} \underset{\sim}{r}(t) = \underset{\sim}{L} = (L_1, L_2, L_3)$$

means as t gets close to c, $\underset{\sim}{r}(t)$ gets close to $\underset{\sim}{L}$,

which means $r_1(t) \longrightarrow L_1$, $r_2(t) \longrightarrow L_2$ and $r_3(t) \longrightarrow L_3$

so $$\lim_{t \to c} \underset{\sim}{r}(t) = \left(\lim_{t\to c} r_1(t), \lim_{t \to c} r_2(t), \lim_{t \to c} r_3(t) \right)$$

limits of functions of two variables

$$f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \underset{\sim}{c} = (c_1, c_2) \in D$$

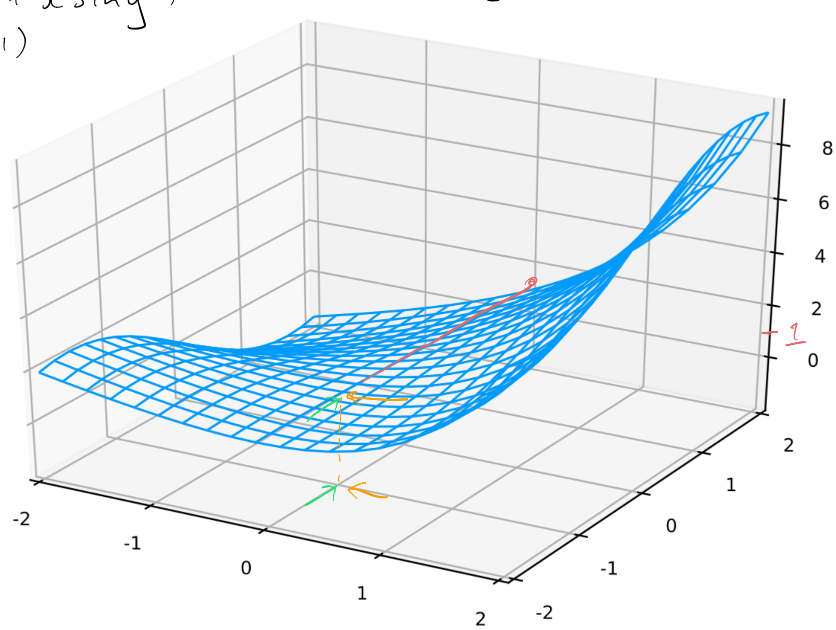
$$\lim_{\underset{\sim}{x} \to \underset{\sim}{c}} f(\underset{\sim}{x}) = \lim_{(x,y) \to (c_1, c_2)} f(x,y) = L$$

means as (x,y) gets close to the point (c_1, c_2);

$f(x,y)$ gets close to L

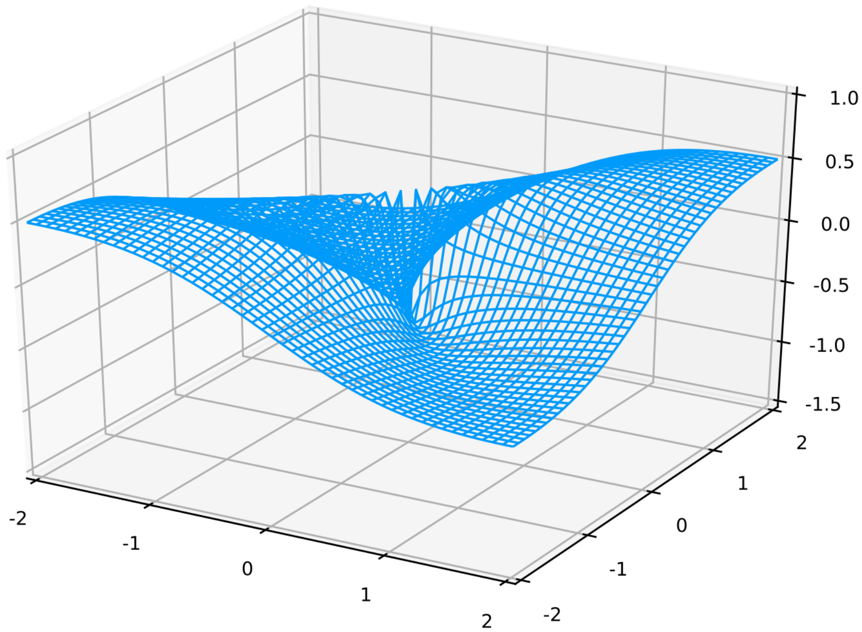
Example

$$\lim_{(x,y) \to (0,-1)} (e^x + x \sin y) = e^0 + 0 = 1$$



Example $\quad f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \quad f(x,y) = \dfrac{xy}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$



More formally: $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{c}} f(\underset{\sim}{x}) = L$ requires that for every curve $\underset{\sim}{r}(t)$ with $\lim\limits_{t \to t_0} \underset{\sim}{r}(t) = \underset{\sim}{c}$, $\quad \lim\limits_{t\to t_0} f(\underset{\sim}{r}(t)) = L$

for the above example, consider $\underset{\sim}{r}_1(t) = (t,t)$, $\quad \underset{\sim}{r}_1(t) \xrightarrow[t\to 0]{} (0,0)$

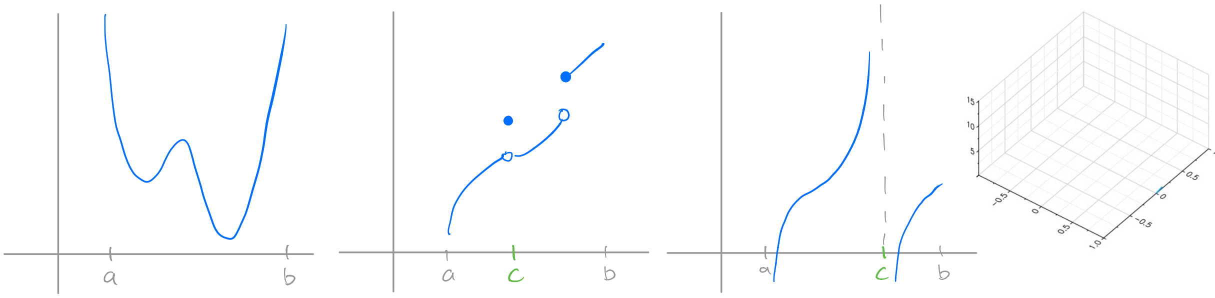
$$f(\underset{\sim}{r}_1(t)) = f(t,t) = \frac{t^2}{t^2+t^2} = \frac{1}{2} \xrightarrow[t\to 0]{} \frac{1}{2}$$

compare this with $\underset{\sim}{r}_2(t) = (t,0) \quad \left(\xrightarrow[t\to 0]{} (0,0) \right)$

$$f(\underset{\sim}{r}_2(t)) = 0 \xrightarrow[t\to 0]{} 0 \neq \frac{1}{2}$$

so the limit doesn't exist.

Recall: a function $f:(a,b) \to \mathbb{R}$ is continuous on (a,b) if you can draw its graph without taking your pen off the page



continuous — not conts. at c — not conts. at c — conts

the points of discontinuity are points c where $\lim_{x \to c} f(x)$ doesn't exist or where $\lim_{x \to c} f(x) \neq f(c)$

So we say f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

f is continuous if it is continuous at every point in its domain.

this definition extends easily to multivariable functions
(defining continuity in terms of drawing the graph does not!)

Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$, $\underset{\sim}{c} \in D$, then f is continuous at $\underset{\sim}{c}$

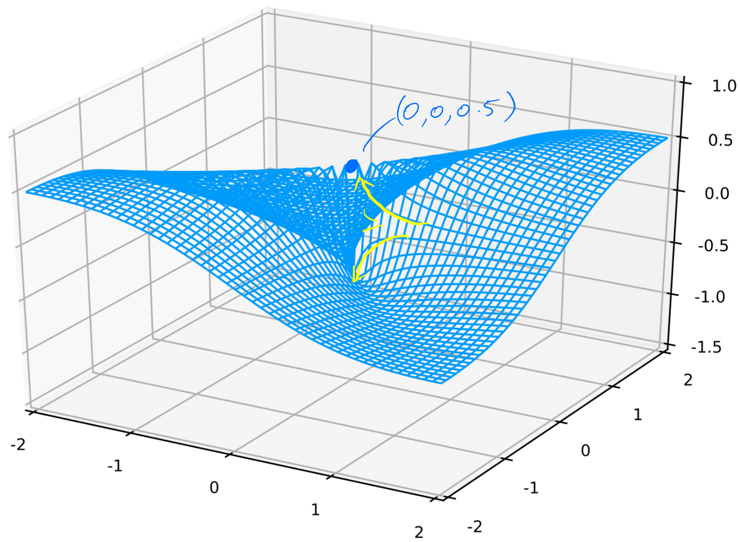
if $\lim_{\underset{\sim}{x} \to \underset{\sim}{c}} f(\underset{\sim}{x}) = f(\underset{\sim}{c})$ i.e. $\lim_{(x,y) \to (c_1,c_2)} f(x,y) = f(c_1,c_2)$

Example $f(x,y) = \dfrac{xy}{x^2+y^2}$ is not defined at $(0,0)$

define

$$g(x,y) = \begin{cases} 0.5 & \text{if } (x,y) = (0,0) \\ \dfrac{xy}{x^2+y^2} & \text{otherwise} \end{cases}$$

is g continuous?



intuitively: not continuous at $\underset{\sim}{0}$ - there is tearing

formally: not continuous: $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist so $\neq f(0,0)$.

for a vector valued function $\underset{\sim}{r}(t)$, the condition for continuity at c

$$\lim_{t\to c} \underset{\sim}{r}(t) = \underset{\sim}{r}(c)$$

implies:

$$\left(\lim_{t\to c} r_1(t), \lim_{t\to c} r_2(t), \lim_{t\to c} r_3(t)\right) = \left(r_1(c), r_2(c), r_3(c)\right)$$

i.e. each coordinate function is continuous.

STANDARD CONTINUOUS FUNCTIONS

polynomials

$P : \mathbb{R} \to \mathbb{R}$, $\quad P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \quad$ where $a_i \in \mathbb{R}$ are fixed.

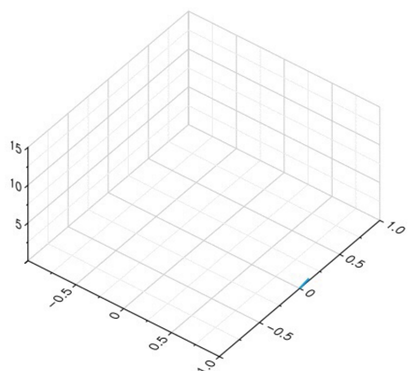
$P(x), Q(x)$ polynomials, $\quad \dfrac{P(x)}{Q(x)}$ is continuous except where $Q(x) = 0$

$\sin x$ and $\cos x$ are continuous on $\mathbb{R}$

e^x is continuous on $\mathbb{R}$

$\ln x$ is continuous on its domain $(0, \infty)$

Examples $\quad \underset{\sim}{r}(t) = (\cos t, \sin t, t)$



$r_1(t) = \cos t \quad$ continuous on $\mathbb{R}$

$r_2(t) = \sin t \quad$ "

$r_3(t) = t \quad$ "

therefore $\underset{\sim}{r}(t)$ is continuous on $\mathbb{R}$

$f : \mathbb{R}^2 \to \mathbb{R}$, $\quad f(x,y) = x$

$$\lim_{(x,y)\to(c_1,c_2)} f(x,y) = \lim_{x \to c_1} x = c_1 \quad \text{because } x \text{ is continuous}$$

$$= f(c_1, c_2)$$

so f is continuous on $\mathbb{R}^2$

Some properties of continuous multivariable functions

If $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$ are both continuous at $\underset{\sim}{c} \in D$ then

1. $a f(\underset{\sim}{x}) + b g(\underset{\sim}{x})$, $a, b \in \mathbb{R}$, and $f(\underset{\sim}{x}) g(\underset{\sim}{x})$ are continuous at $\underset{\sim}{c}$

2. if $g(\underset{\sim}{c}) \neq 0$ then $\dfrac{f(\underset{\sim}{x})}{g(\underset{\sim}{x})}$ is continuous at $\underset{\sim}{c}$

(see Theorem 2.22 in the course reader)

Example

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$(x,y) \mapsto x$ is continuous on $\mathbb{R}^2$ (see above)

by similar arguments $(x,y) \mapsto y$, $(x,y) \mapsto x^2$, $(x,y) \mapsto y^2$ are all continuous on $\mathbb{R}^2$, so applying properties 1. and 2. above we have that $f(x,y) = \dfrac{xy}{x^2+y^2}$ is continuous on $\mathbb{R}^2$ except at $(0,0)$.