The limit at a point (intuitive definition)

we say the limit as x approaches c of $f(x)$ is L

notation:

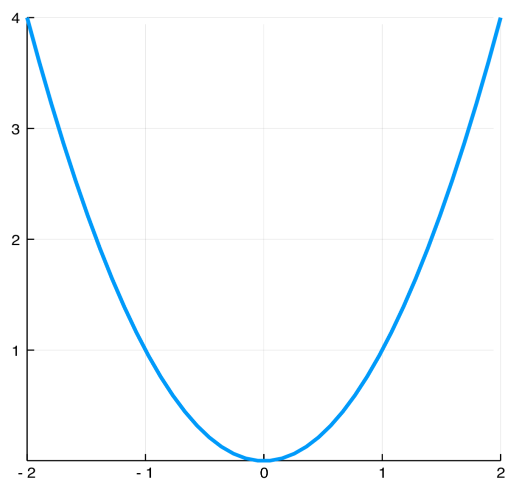
if when x is close to c $f(x)$ is close to L,
and we can make $f(x)$ as close as we like to L by taking x sufficiently close to c.

Examples

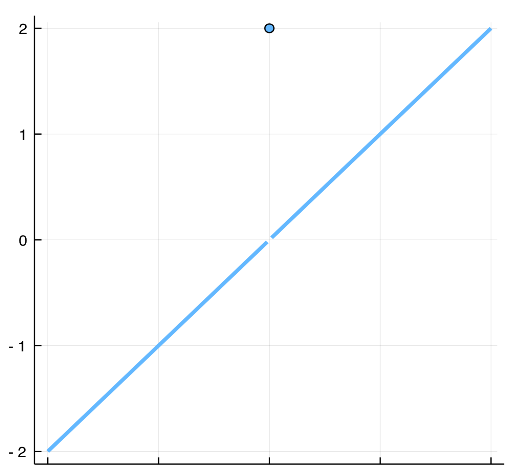
$f(x) = x^2$

$\lim_{x \to 0} f(x)$

$\lim_{x \to \sqrt{2}} f(x)$

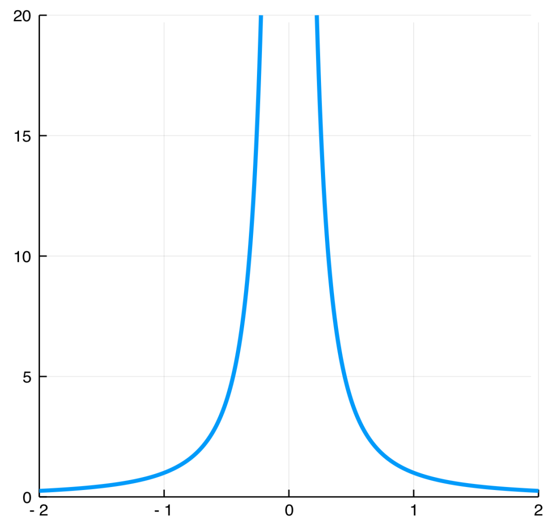


$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \{$ (piecewise function)



$\lim_{x \to 0} f(x) =$

$f(x) = \frac{1}{x^2}$



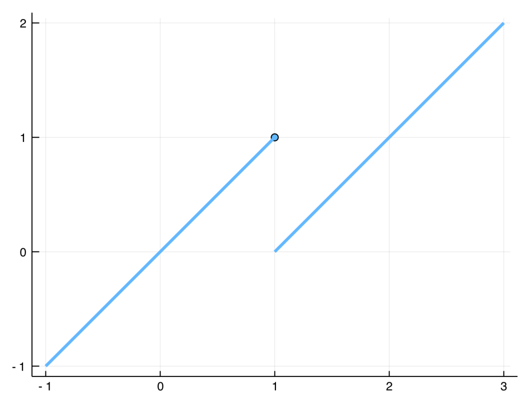
this is an example of a function diverging to infinity:
we say f diverges to infinity at c if we can make $f(x)$ as large as we like by taking x sufficiently close to c

notation:

remember though:

$f(x) = \begin{cases} \\ \end{cases}$

$\lim_{x \to 1} f(x) =$



one sided limits:

$x \to 1$ with $x < 1$

$x \to 1$ with $x > 1$

$\lim_{x \to c} f(x)$ exists $\Leftrightarrow$ both one-sided limits exist and are equal.

the limit as $x \to \infty$

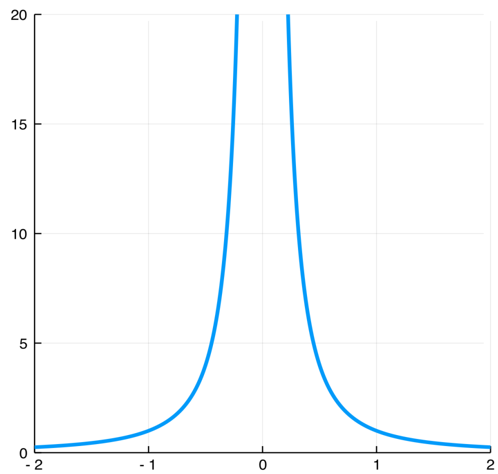
we write if we can make $f(x)$ as close

to L as we like by taking x sufficiently large.

similar.

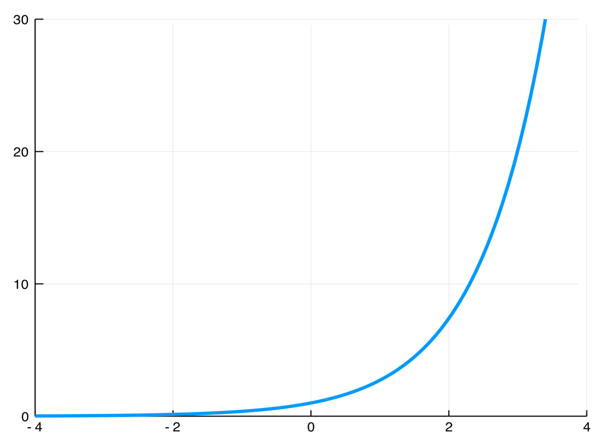
Examples

$f(x) = \frac{1}{x^2}$



in fact: $\lim_{x \to \infty} \frac{1}{x^p} = 0$ for any constant $p > 0$

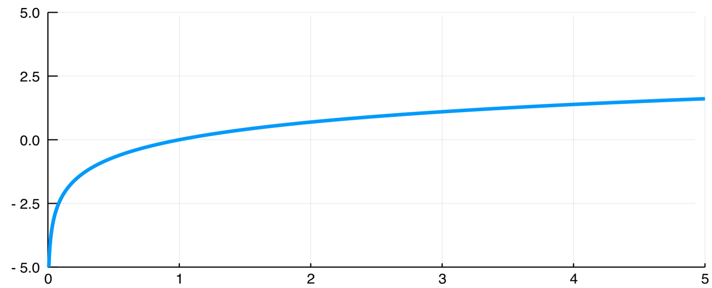
$f(x) = e^x$



$f(x) = \ln x$

$$\lim_{x \to 0^+} \ln x$$

$$\lim_{x \to \infty} \ln x$$



Limit laws

If f and g are functions $(b,c) \to \mathbb{R}$, $a \in (b,c)$ and the limits

$\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$ both exist, then

- $\lim_{x \to a} \qquad = \lim_{x \to a} \qquad + \lim_{x \to a}$

- for any constant $k \in \mathbb{R}$

  $\lim_{x \to a} \qquad = \lim_{x \to a}$

- $\lim_{x \to a} \qquad) = \lim_{x \to a} \qquad . \lim_{x \to a}$

- if $\lim_{x \to a} g(x) \neq 0$, then

  $$\lim_{x \to a} \qquad = \frac{\lim_{x \to a}}{\lim_{x \to a}}$$

These laws also hold for one-sided limits $\lim_{x \to a^+}$, $\lim_{x \to a^-}$

and limits at infinity $\lim_{x \to \pm \infty}$

Examples

1. $\lim_{x \to 2} (x^2 + 2x) =$

2. $\lim_{x \to \pi} \dfrac{x^2}{\cos x}$ $\qquad \lim_{x \to \pi} x^2$, $\lim_{x \to \pi}$

therefore

3. $\lim_{x \to -2} \dfrac{x^2+2x}{x+2}$

4. $\lim_{x \to \infty} 1-2x$ $\lim_{x \to \infty} 1+2x$

$\lim_{x \to \infty} (1-2x + 1+2x)$

5. $\lim_{x \to \infty} \dfrac{x^2+2x+1}{2x^2+x+7} = \lim_{x \to \infty}$

numerator:

$\lim_{x \to \infty}$

denominator:

$\lim_{x \to \infty}$

denominator $\neq 0$:

$\lim_{x \to \infty} \dfrac{x^2+2x+1}{2x^2+x+7}$

The squeeze theorem

Let $f, g, h : (a, b) \to \mathbb{R}$ and suppose

$$f(x) \le g(x) \le h(x) \qquad \text{for all } x \in (a, b)$$

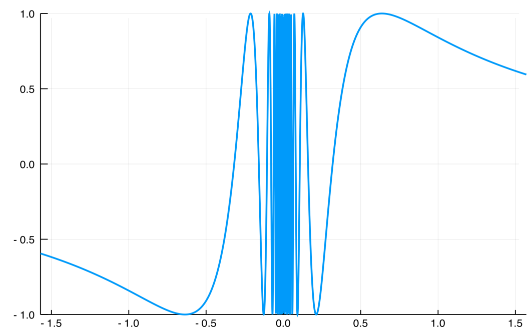
then

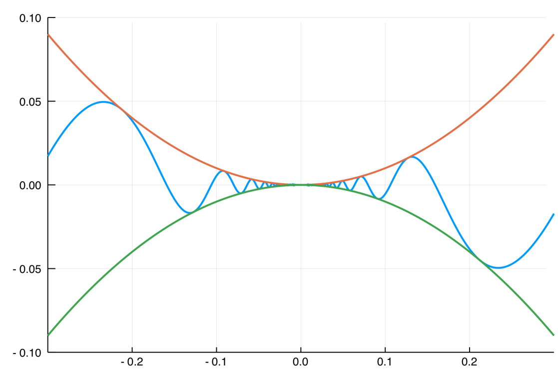
Example $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

$\lim_{x \to 0} x^2$

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

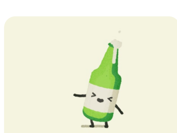
but x^2 dominates:





so by the squeeze theorem

- a.k.a. "two policemen and a drunk" theorem



$\sin \frac{1}{x}$

$x^2, \; -x^2$

The theorem also holds for limits at infinity $\lim_{x \to \infty}$ or $\lim_{x \to -\infty}$

Example $\lim_{x \to \infty} \frac{2\cos x}{x}$

$\lim_{x \to \infty} \frac{2\cos x}{x}$ by the squeeze theorem.

First recall the intuitive definition:

if when x is close to c $f(x)$ is close to L,
and we can make $f(x)$ as close as we like to L by taking x sufficiently close to c.

what exactly is meant by "close" and "as close as we like"?

The precise definition of a limit at a point:

if for all there exists such that

if then

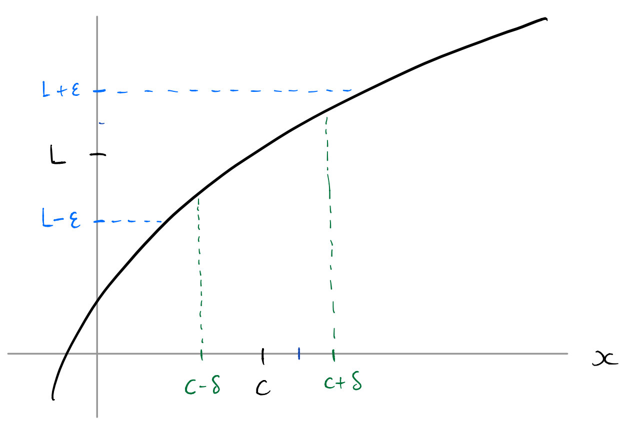
i.e.

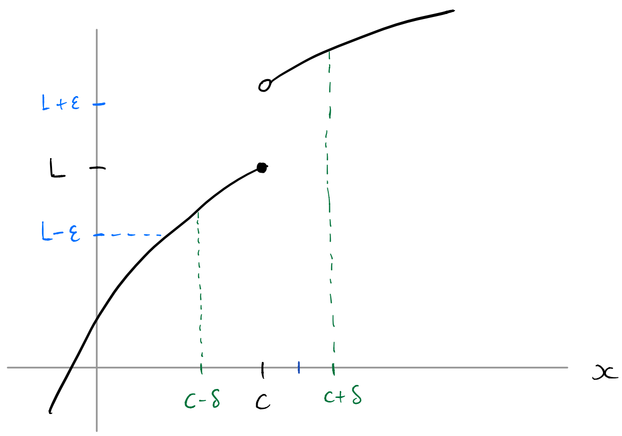
if then

for ε, δ small : "x is close to c" "$f(x)$ is close to L"

since it is <u>for all</u> ε, ε can be as small as we like and a δ still exists — this captures the "as close as we like" part.

graphically





Example $f(x) = 2x$

expect – let's prove it

need to show:

Since we need to prove that for all ε there exists δ,

we must find an expression for δ

As a first guess:

if then

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

this is close, but not quite what is needed ( )

Can fix it by letting instead:

if then

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

therefore, for all $\varepsilon > 0$ there exists δ (for example)
such that
i.e.

$\square$

Definition of the limit at infinity

Given $f: (c, \infty) \to \mathbb{R}$ write $\lim_{x \to \infty} f(x) = L$ if for all $\varepsilon > 0$ there exists b such that if $x \geq b$ then $|f(x) - L| < \varepsilon$

Definition of diverging to infinity

$\lim_{x \to a^-} f(x) = \infty$ if for any $M > 0$ there exists $\delta > 0$ such that if $x < a$ and $|x - a| < \delta$ then $f(x) > M$.

Diverging to infinity at infinity

$\lim_{x \to \infty} f(x) = \infty$ if for any $M > 0$ there exists b such that if $x \geq b$ then $f(x) > M$.

Example $\lim_{x \to \infty} \ln x$

proof: given $M > 0$, let b

Note that: for all $x > 0$ so $\ln x$ is an increasing function. Therefore taking logs preserves the inequality:

$x \geq b \quad \Rightarrow$

limits of vector valued functions

means as t gets close to c, $\underset{\sim}{r}(t)$ gets close to $\underset{\sim}{L}$,

which means

so $\displaystyle\lim_{t \to c} \underline{r}(t) =$

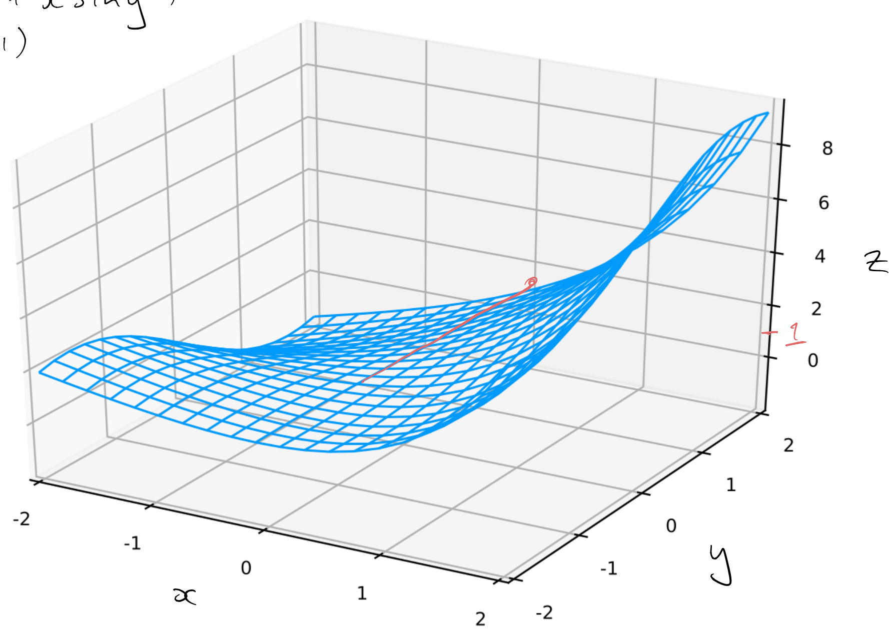
limits of functions of two variables

means as gets close to the point

gets close to

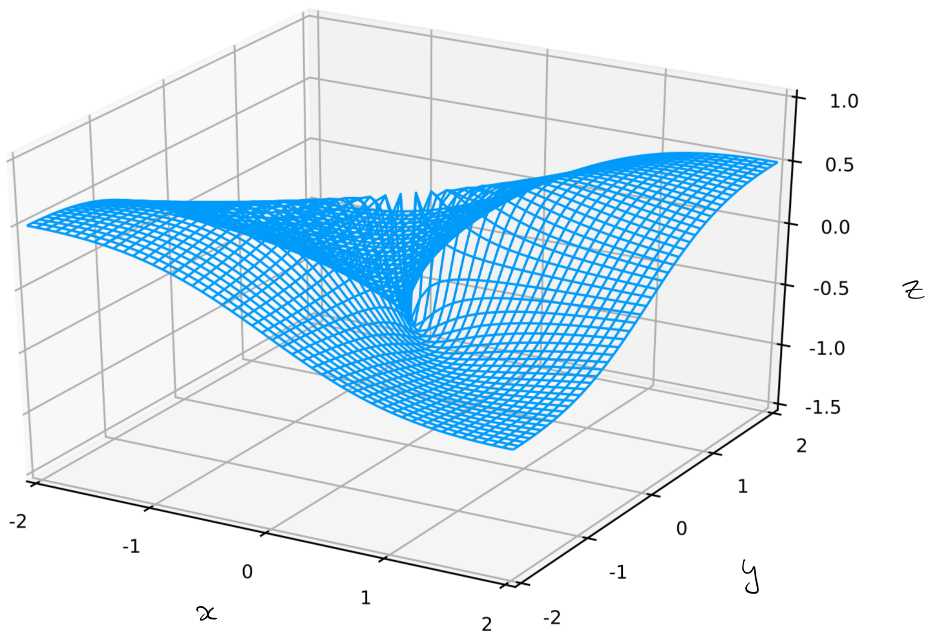
Example

$$\lim_{(x,y) \to (0,-1)} (e^x + x \sin y) =$$



Example $\quad f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \quad f(x,y) = \dfrac{xy}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$



More formally: $\lim_{\underset{\sim}{x}\to\underset{\sim}{c}} f(\underset{\sim}{x}) = L$ requires that for every curve

$\underset{\sim}{r}(t)$ with ,

for the above example, consider

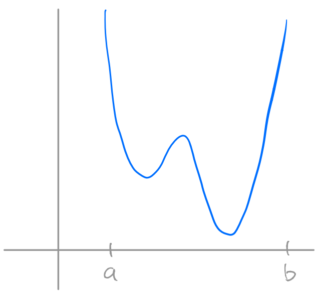
$f(\underset{\sim}{r}_1(t)) =$

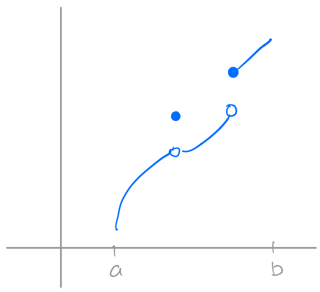
compare this with

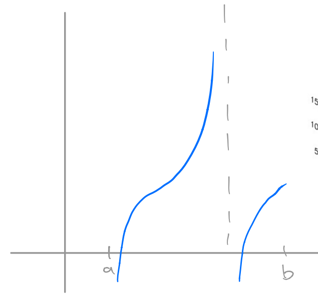
$f(\underset{\sim}{r}_2(t)) =$

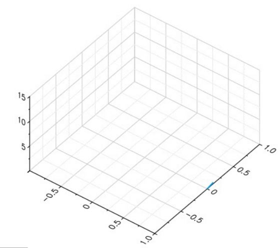
so the limit doesn't exist.

Recall: a function $f:(a,b) \to \mathbb{R}$ is continuous on (a,b) if you can draw its graph without taking your pen off the page









conts

the points of discontinuity are points where doesn't exist or where $\lim_{x \to c} f(x)$

So we say f is continuous at c if

f is continuous if it is continuous at every point in its domain.

this definition extends easily to multivariable functions (defining continuity in terms of drawing the graph does not!)

Let $f:$, , then f is continuous at $\underset{\sim}{c}$

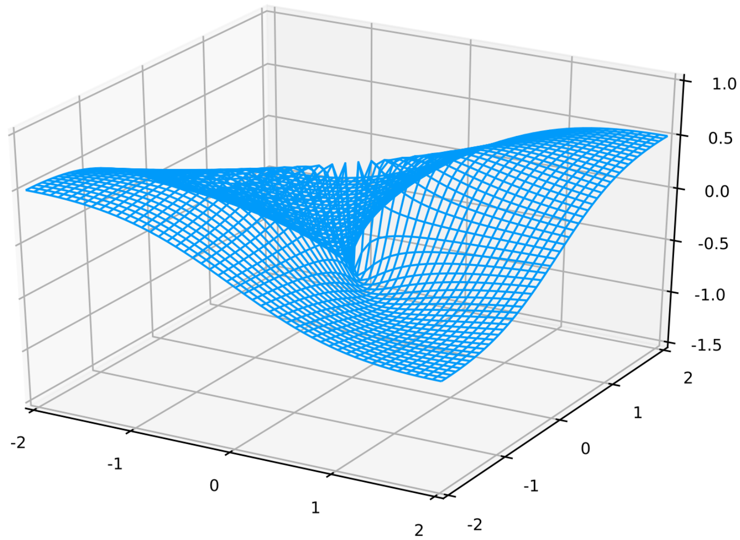
if i.e.

Example $f(x,y) = \dfrac{xy}{x^2+y^2}$ is not defined at $(0,0)$

define

$$g(x,y) = \begin{cases} 0.5 & \text{if } (x,y) = (0,0) \\ \dfrac{xy}{x^2+y^2} & \text{otherwise} \end{cases}$$

is g continuous?



intuitively: not continuous at $\underset{\sim}{0}$ - there is tearing

formally: not continuous: $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist so $\neq f(0,0)$.

for a vector valued function $\underset{\sim}{r}(t)$, the condition for continuity at c

implies:

$$\left(\lim_{t\to c} r_1(t), \lim_{t\to c} r_2(t), \lim_{t\to c} r_3(t)\right) = \left(r_1(c), r_2(c), r_3(c)\right)$$

i.e. each coordinate function is continuous.

STANDARD CONTINUOUS FUNCTIONS

polynomials

$P : \mathbb{R} \to \mathbb{R}$, $\quad P(x) =$ $\qquad$ where $a_i \in \mathbb{R}$ are fixed.

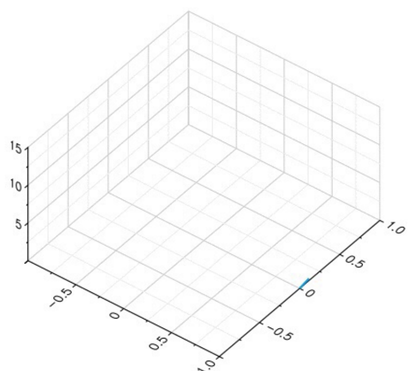
$P(x), Q(x)$ polynomials, $\dfrac{P(x)}{Q(x)}$ is continuous except where $Q(x) = 0$

and $\qquad$ are continuous on $\mathbb{R}$

is continuous on $\mathbb{R}$

is continuous on its domain

Examples $\quad \underset{\sim}{r}(t) = (\cos t, \sin t, t)$



$r_1(t) = \cos t \qquad$ continuous on $\mathbb{R}$

$r_2(t) = \sin t \qquad$ "

$r_3(t) = t \qquad$ "

therefore $\underset{\sim}{r}(t)$ is continuous on $\mathbb{R}$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\quad f(x,y) = x$

so f is continuous on $\mathbb{R}^2$

Some properties of continuous multivariable functions

If $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$ are both continuous at $\underset{\sim}{c} \in D$ then

1. $a f(\underset{\sim}{x}) + b\, g(\underset{\sim}{x})$, $a, b \in \mathbb{R}$, and $f(\underset{\sim}{x})\, g(\underset{\sim}{x})$ are continuous at $\underset{\sim}{c}$

2. if $g(\underset{\sim}{c}) \neq 0$ then $\dfrac{f(\underset{\sim}{x})}{g(\underset{\sim}{x})}$ is continuous at $\underset{\sim}{c}$

(see Theorem 2.22 in the course reader)

Example

$$f(x,y) = \frac{xy}{x^2+y^2}$$

is continuous on $\mathbb{R}^2$ (see above)

by similar arguments
are all continuous on $\mathbb{R}^2$, so applying properties 1. and 2. above we have that $f(x,y) = \dfrac{xy}{x^2+y^2}$ is continuous on $\mathbb{R}^2$ except at $(0,0)$.