$f : I \to \mathbb{R}$ a function defined on an interval $I \subset \mathbb{R}$

f is differentiable at $c \in I$ if the limit

exists

this limit is called the derivative of f at c, denoted:

If f is differentiable at every $c \in I$, then

is a function

equivalent definition: (change of variable)

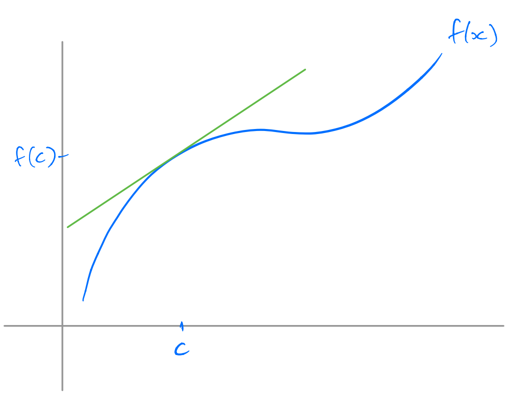
therefore $f'(c) =$

Intuition: if $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c)$

then when x is close to c, $\dfrac{f(x) - f(c)}{x - c}$ is close to $f'(c)$

i.e:

rearranging: $f(x) \approx$



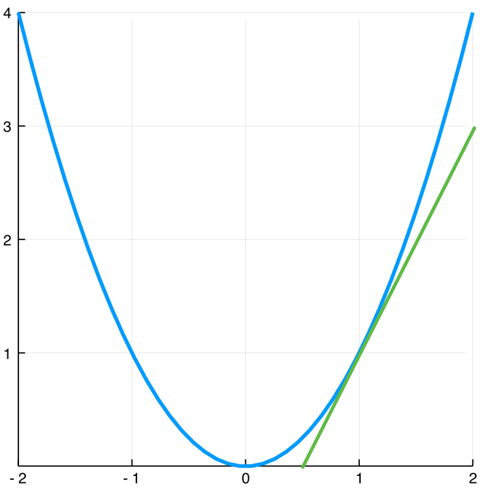
this is the equation for a line with slope $f'(c)$ passing through the point $(c, f(c))$

Example $\quad f(x) = x^2$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} =$$

$$=$$

$$=$$



$f'(1) =$

so when $x \approx 1$:

$f(x) \approx$

$=$

$=$

i.e. $y = 2x - 1$ is the tangent line to x^2 at $x = 1$.

Differentiation rules

If $f, u, v : I \to \mathbb{R}$ are differentiable functions then

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

$c \in \mathbb{R}$: $\quad \frac{d}{dx}(cf) = c\,\frac{df}{dx}$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$ product rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ quotient rule

$$\frac{d}{dx} f(u(x)) = \frac{df}{du}\frac{du}{dx}$$ chain rule

Functions take an input and assign a single output

.

They can be many-to-one, meaning two different inputs

, might have the same output:

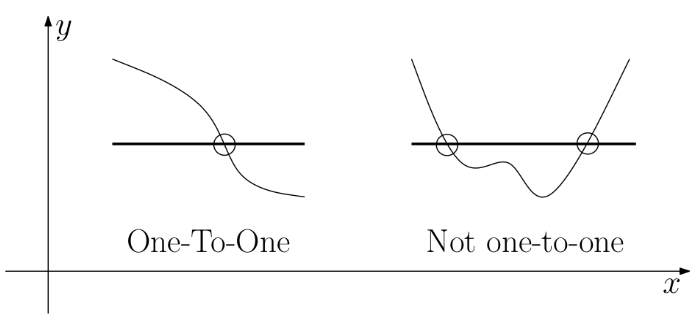
e.g:

Or they can be one-to-one meaning that distinct inputs always have distinct outputs:

eg:

then

the horizontal line test:



A natural question: given an output y, what was the input?

i.e.

- If f is many-to-one there can be multiple solutions

- If f is one-to-one then there is only <u>one</u> answer to this question and therefore the operation

$$\text{output} \longmapsto \text{corresponding input}$$

is a function.

It is called the inverse function and denoted

Note:

Example

Properties of inverse functions

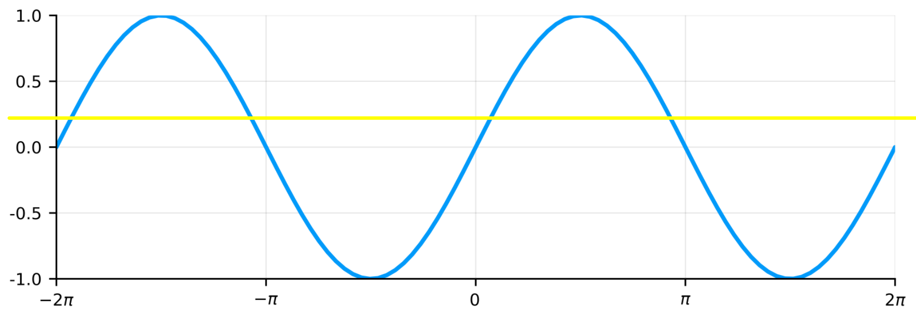
domain f^{-1} = (because f^{-1} takes outputs of f as its inputs)

$f^{-1}(f(x))$ = f^{-1} "undoes" whatever f did

$f(f^{-1}(x))$ = f " " f^{-1} "

$(f^{-1})'(x)$ =

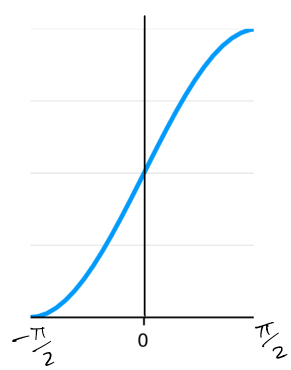
Inverse Function Theorem



this function is not one-to-one. However, if we restrict the domain:

$f:$

then the function is one-to-one:



so an inverse function exists.

Notation:

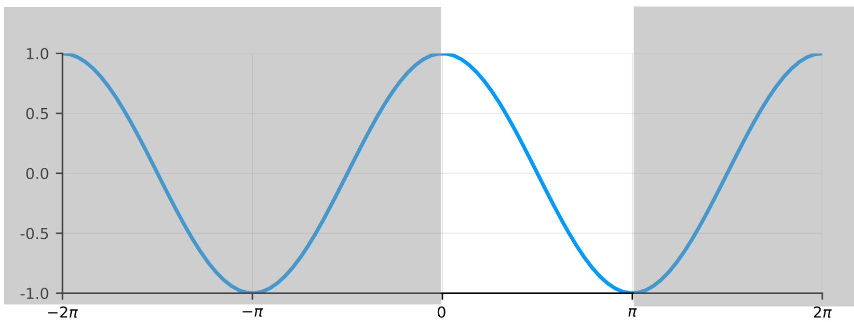
If $\sin x = y$ then

remember the domain of f^{-1} is the range of f, so

$\sin^{-1}: \quad \longrightarrow$

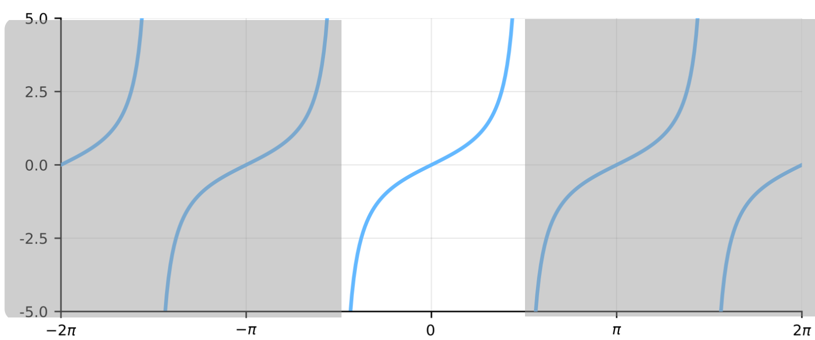
Remark: sin is also one-to-one when restricted to other intervals, eg: $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, but it is conventional to use the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

For $\cos x$ the convention is to restrict to



So $\cos^{-1}$:

$\tan x$ is restricted to



and $\tan^{-1}$:

Derivatives

Inverse function theorem:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{when } f'(x) \neq 0$$

therefore if $f(x) = \sin x$

This can be simplified

therefore

$$\frac{d}{dx}(\sin^{-1})(x) =$$

by similar methods:

$$\frac{d}{dx}(\cos^{-1})(x) =$$

$$\frac{d}{dx}(\tan^{-1})(x) =$$

you will need to memorise these derivatives.

$\underset{\sim}{r} : I \to \mathbb{R}^n \qquad \underset{\sim}{r}(t) =$

$t_0 \in I$, define

$$\underset{\sim}{r}'(t_0) =$$

← this is a vector!

recalling that for any vector valued function

$$\underset{\sim}{f}(t) = (f_1(t), \ldots, f_n(t))$$

it follows that

$\underset{\sim}{r}'(t) =$

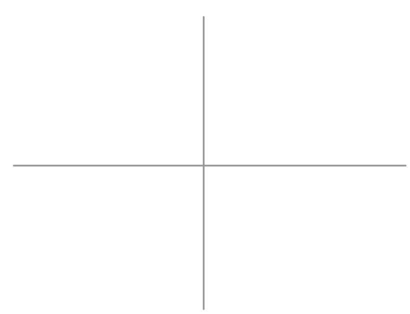
$\underset{\sim}{r}'(t) =$

Example $\quad \underset{\sim}{r} : [0, 2\pi) \longrightarrow \mathbb{R}^2, \quad \underset{\sim}{r}(t) = (\cos t, \sin t)$

$t = 0$

$t = \frac{\pi}{4}$

$t = \frac{\pi}{2}$



→ tangent vector! (vector in the direction of tangent line) if $\underset{\sim}{r}'(t_0) = \lim_{t \to t_0} \frac{\underset{\sim}{r}(t) - \underset{\sim}{r}(t_0)}{t - t_0}$

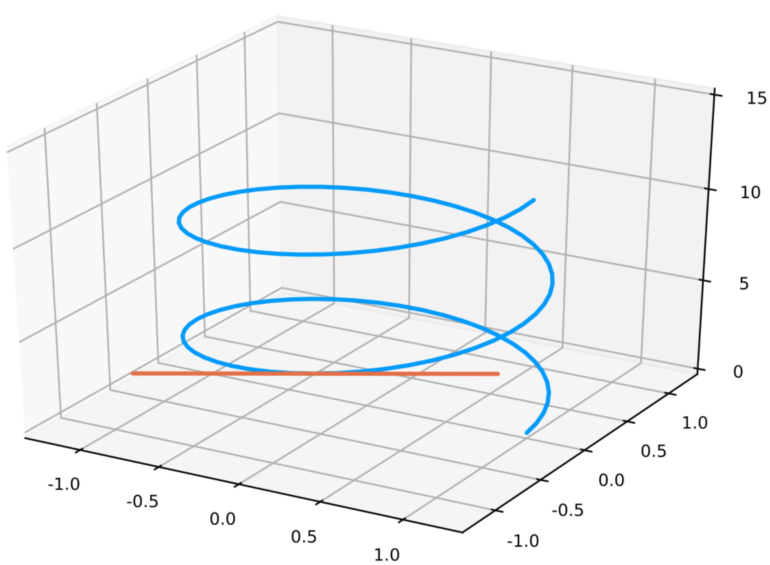
then when t is close to t_0:

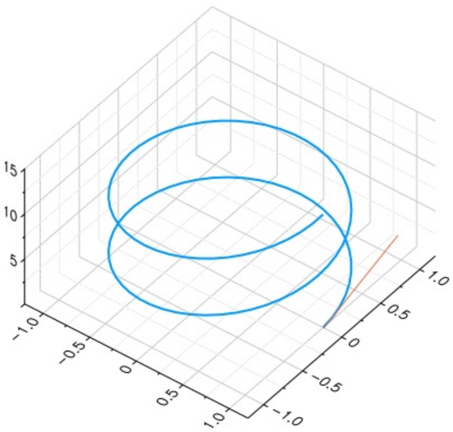
rearranging:

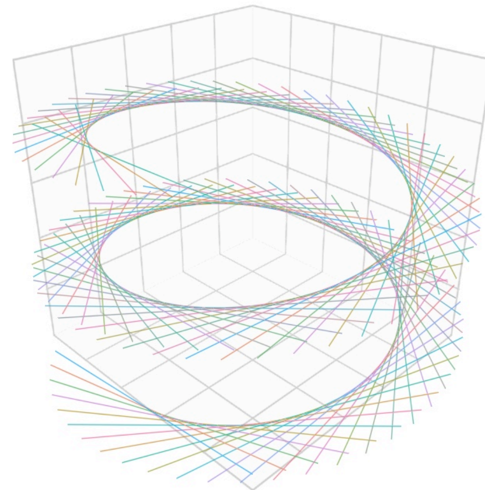
this is the line that approximates $\underset{\sim}{r}(t)$ at $t = t_0$,
i.e. the tangent line.
$\underset{\sim}{r}'(t_0)$ is the direction vector of the tangent line at $\underset{\sim}{r}(t_0)$.

Example $\underset{\sim}{r}(t) = (\cos t, \sin t, t)$

tangent lines: plotted below:







Typical application: $\underset{\sim}{r}(t)$ is the position at time t of an object moving through space.

the trajectory of the object is the curve

the velocity at time t is the rate of change of position

the speed at time t is the magnitude of velocity

the acceleration at time t is the rate of change of velocity

Newton's second law of motion

force acting on an object with mass m

$\underset{\sim}{u} : I \to \mathbb{R}^n$ $\underset{\sim}{v} : I \to \mathbb{R}^n$ differentiable vector valued functions

$\frac{d}{dt}\left(\underset{\sim}{u}(t) + \underset{\sim}{v}(t)\right) =$

$\frac{d}{dt}\left(c\,\underset{\sim}{u}(t)\right) =$ for any $c \in \mathbb{R}$

Product rules $f : I \to \mathbb{R}$

$\frac{d}{dt}\left(f(t)\,\underset{\sim}{u}(t)\right) =$

$\frac{d}{dt}\,\underset{\sim}{u}(t) \cdot \underset{\sim}{v}(t) =$

Chain rule

If $\alpha : I \to I$ then

$\frac{d}{dt}\,\underset{\sim}{u}(\alpha(t))$

Examples

$\underset{\sim}{c}(t) = (\cos t, \sin t)$

Let $\alpha(t) =$ and $\underset{\sim}{r}(t) =$

then $\underset{\sim}{r}'(t) =$

$=$

Compare speeds:

$|\underset{\sim}{c}'(t)| =$

$|\underset{\sim}{r}'(t)| =$

$=$

Find $\frac{d}{dt} |\underset{\sim}{c}(t)|^2 =$

by the (dot) product rule:

$=$

$=$

$=$

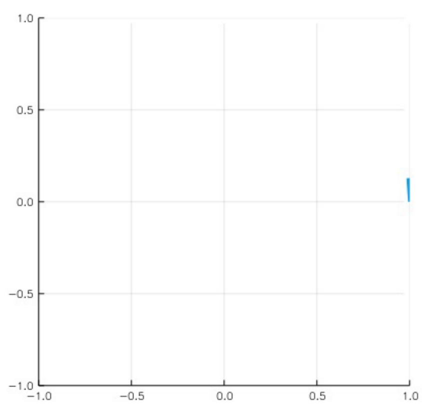
$=$

$=$

this is expected because $\underset{\sim}{c}(t)$ is a parametrization of the unit circle $\longrightarrow$

therefore

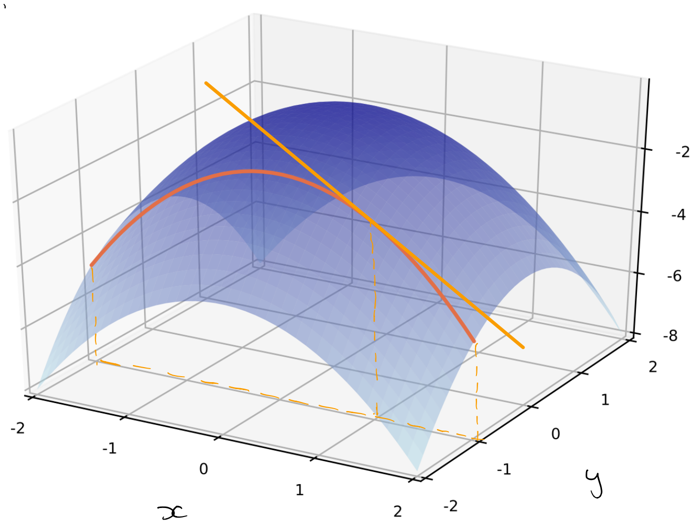
notice also that , i.e. the tangent vector $c'(t)$ is perpendicular to the position $\underset{\sim}{c}(t)$



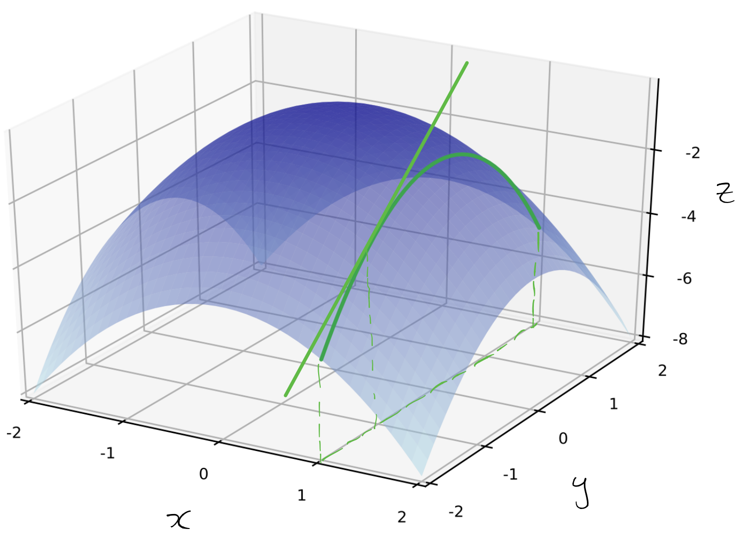
Partial derivatives

how can we find tangent vectors / lines to a surface?

eg:



the slope at $(x, -1)$ in the x direction is



the slope at $(1, y)$ in the y direction is

These slopes are called partial derivatives

Notation

partial derivative of f with respect to x at $(1,-1)$

partial derivative of f with respect to y at $(1,-1)$

Formally:

$\frac{\partial}{\partial x} f(a,b) =$ $\qquad$ y fixed $= b$

$\frac{\partial}{\partial y} f(a,b) =$ $\qquad$ x fixed $= a$

In practice this means that to find:

$\frac{\partial f}{\partial x}(a,b)$ take the derivative of $f(x,y)$ wrt $\boldsymbol{x}$ while treating $\boldsymbol{y}$ as though it were a constant, and then substitute $x = a$, $y = b$

$\frac{\partial f}{\partial y}(a,b)$ take the derivative of $f(x,y)$ wrt $\boldsymbol{y}$ while treating $\boldsymbol{x}$ as though it were a constant, and then substitute $x = a$, $y = b$

Examples

$f(x,y) = -x^2 - y^2$, find $\frac{\partial f}{\partial x}(1,-1)$

$g(x,y) = x^2 + xy + y^2$, find $\frac{\partial g}{\partial y}(1,1)$

Find $\frac{\partial g}{\partial x}(a,b)$:

$\frac{\partial g}{\partial x}(x,y)$

All of this can be extended to functions $f : \mathbb{R}^3 \to \mathbb{R}$

or even $\mathbb{R}^n \to \mathbb{R}^m$

(we just can't visualise it as well)

eg: $\frac{\partial f}{\partial z}(a,b,c) =$

to find $\frac{\partial f}{\partial z}$ pretend <u>both</u> x and y are constants.

Examples

$f(x,y,z) = xye^z + \sin(xy)$, find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$g : \mathbb{R}^4 \to \mathbb{R}$ $\quad \underset{\sim}{x} = (x_1, x_2, x_3, x_4)$

$g(\underset{\sim}{x}) = x_1 x_2 - x_3 x_4$ $\quad$ Find $\frac{\partial g}{\partial x_4}$

There are several common notational alternatives for partial derivatives:

We will mostly use the first three. Sometimes the function's arguments are written: eg: , often they are suppressed:

Second partial derivatives

Suppose has partial derivative

which is differentiable with respect to x. Then

is called the second partial derivative with respect to x.
Alternative notation:

Examples

$g(x,y) = x^2 + xy + y^2$

$f(x,y) = e^{2y} \sin x$

Similarly we define (assuming they exist)

$f_{xy} =$

$f_{yy} =$

$f_{yx} =$

f_{xy} and f_{yx} are called mixed partial derivatives

If the mixed partial derivatives are defined and continous on $\mathbb{R}^2$ then they are equal (Clairaut's / Schwarz' theorem)

Example

$$f(x,y) = e^{2y}\sin x$$

we can also calculate third-order partial derivatives

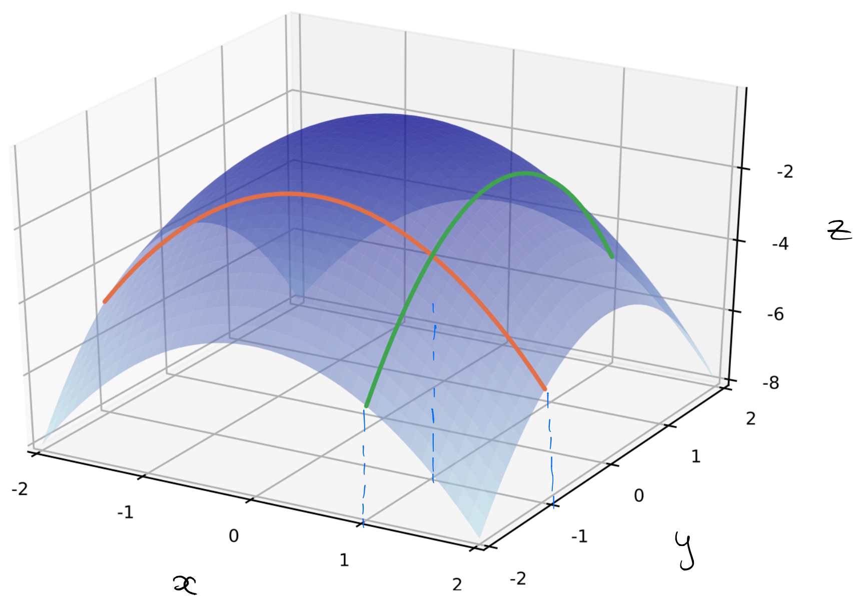
eg: $f_{xxy} =$

and fourth-order ...

and partial derivatives of functions of three or more variables

eg: $F(x,y,z) = xy - xz$

Recall: The partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ give the slopes of tangent lines in the x and y directions respectively



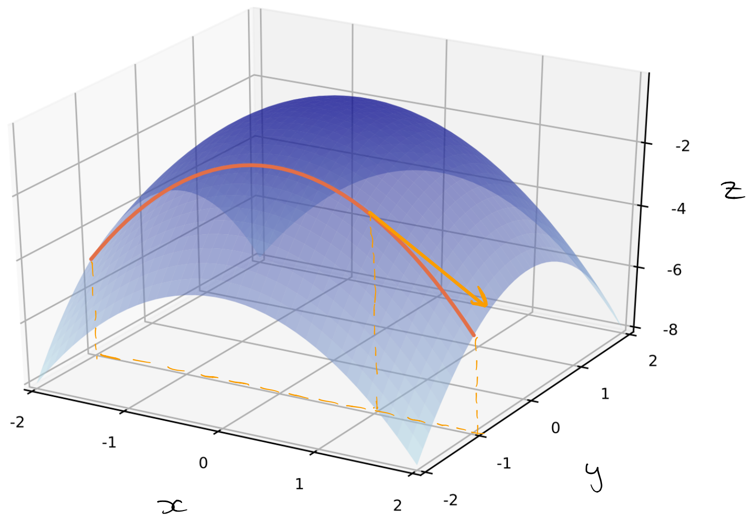
direction vectors for the tangent lines (a.k.a tangent vectors) at $f(1,-1)$ are given by

← for an increase of 1 in the x direction there is an increase/decrease of $\frac{\partial f}{\partial x}(1,-1)$ in the z direction

(slope = $\frac{\text{rise}}{\text{run}}$)

← for an increase of 1 in the y direction there is an increase/decrease of $\frac{\partial f}{\partial y}(1,-1)$ in the z direction

We can also calculate these vectors as follows.



graph f =

fix $y = -1$, this gives the orange curve

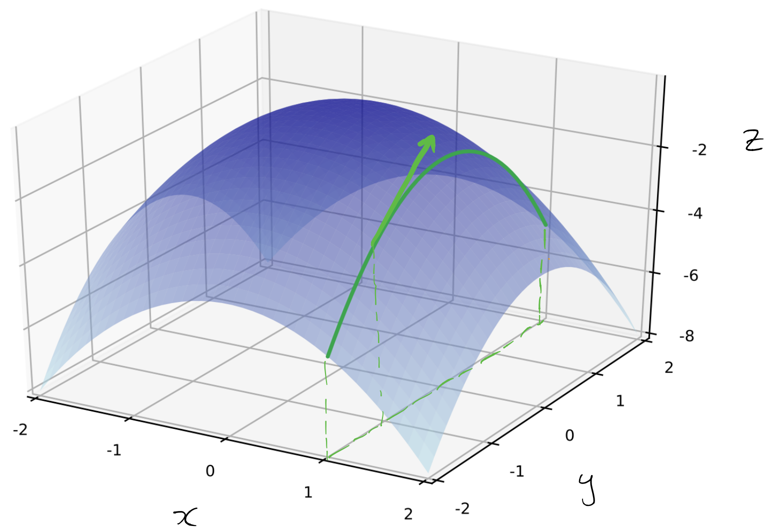
which we can parametrize by $\underset{\sim}{c} : \mathbb{R} \to \mathbb{R}^3$

we find tangent vectors to this curve by taking the derivative with respect to x

For the given example $f(x,y) = -x^2 - y^2$

tangent line in the x-direction at $(1,-1,-2)$:

a tangent vector in the y direction.



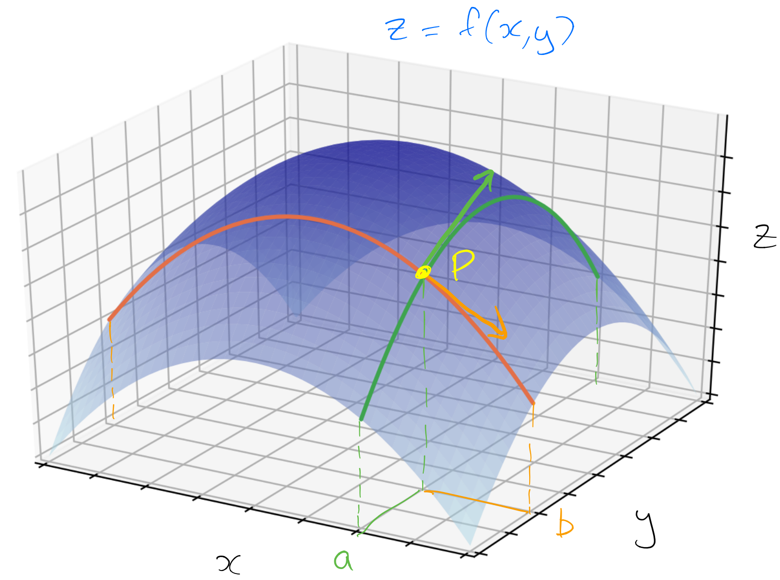
fix $x = 1$ (green curve). Parametrize by

$\underset{\sim}{q}(y) =$

$\underset{\sim}{q}'(y) =$

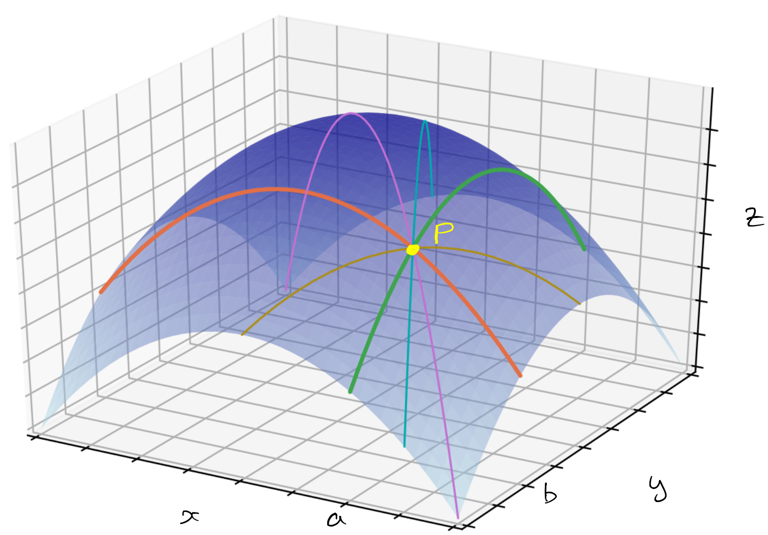
at $y = -1$:

An equation for the tangent line in the y direction at $(1,-1,-2)$



The tangent vectors at $\underline{P} = (a, b, f(a,b))$ are

There are other curves on the surface passing through the point $\underline{P} = (a, b, f(a,b))$:



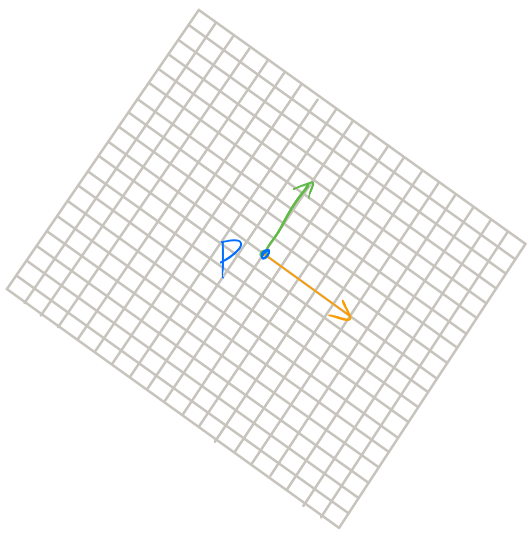
→ there are infinitely many tangent lines (and vectors) at this point, but they all lie in a common plane, called the tangent plane at $\underset{\sim}{P}$

How can we describe this plane mathematically?

recall: equation of the tangent line in x-direction

$$\underset{\sim}{r}(t) =$$

observation: any point in the tangent plane can be obtained as a $\underset{\sim}{P}$ + a sum of scalar multiples of the tangent vectors in the x and y directions



i.e. if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in the tangent plane then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

i.e. $x =$

$y =$

$z =$

parametric equations for tangent plane

we can also characterise the tangent plane by an implicit equation, i.e. by writing z in terms of x and y:

rearranging

substituting into the equation for z:

$z =$

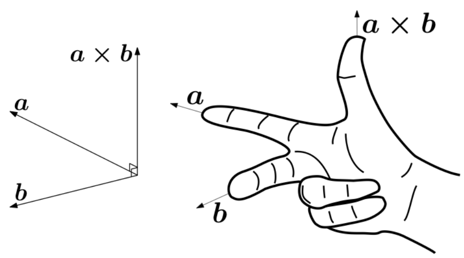
Cross product

The cross product $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$, $\underset{\sim}{a} = (a_1, a_2, a_3)$ $\underset{\sim}{b} = (b_1, b_2, b_3)$

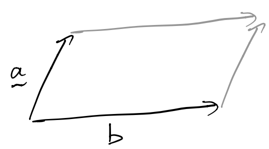
$$\underset{\sim}{a} \times \underset{\sim}{b} = (\qquad , \qquad , \qquad)$$

it has some very useful properties:

- $\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and oriented according to the right hand rule:



- the magnitude $\| \underset{\sim}{a} \times \underset{\sim}{b} \|$ is equal to the area of the parallelogram:



Example $\underset{\sim}{a} = (1, 3, 1)$ $\underset{\sim}{b} = (2, 1, 5)$

$\underset{\sim}{a} \times \underset{\sim}{b} =$

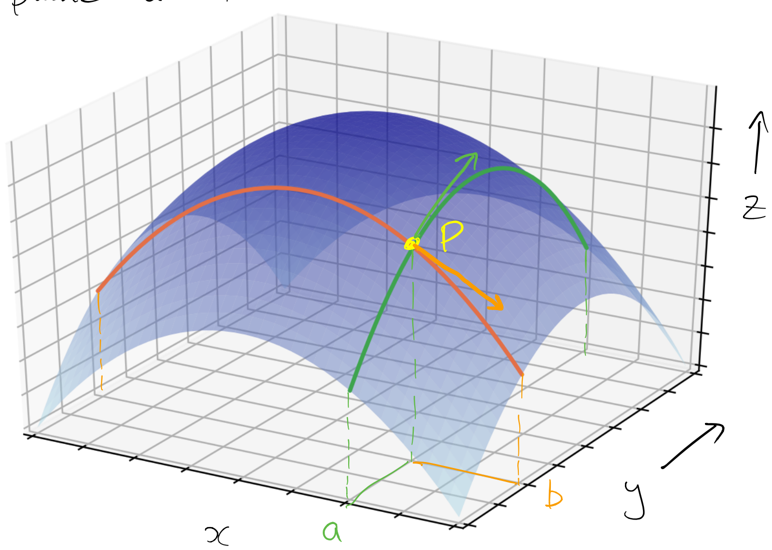
$=$

$\underset{\sim}{a} \cdot (\underset{\sim}{a} \times \underset{\sim}{b}) =$

$=$

$=$

A vector $\underset{\sim}{n}$ is called a normal vector to a given surface at the point P if it is perpendicular to every tangent vector at P, i.e. it is perpendicular (a.k.a. orthogonal) to the tangent plane at P.

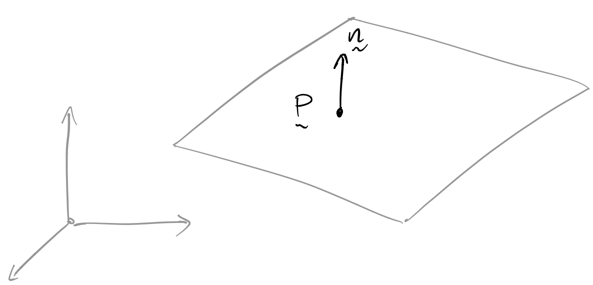


An easy way to find a normal vector is to take the cross product of the tangent vectors in the x and y-directions. Recall that at the point $\underset{\sim}{P} = (a, b, f(a,b))$ these vectors are

$$\underset{\sim}{u} = \left(1, 0, \frac{\partial f}{\partial x}(a,b)\right) \quad \text{and} \quad \underset{\sim}{v} = \left(0, 1, \frac{\partial f}{\partial y}(a,b)\right)$$

$$\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v} = (\qquad\qquad\qquad)$$

A normal vector gives another way of finding an equation for the tangent plane at P:



$\underset{\sim}{P} = (a, b, c)$

$\underset{\sim}{x} = (x, y, z)$

$\underset{\sim}{n} = (n_1, n_2, n_3)$

$\underset{\sim}{x}$ is in the plane orthogonal to $\underset{\sim}{n}$ iff the vector from $\underset{\sim}{P}$ to $\underset{\sim}{x}$, i.e. $\underset{\sim}{x} - \underset{\sim}{P} = (x-a, y-b, z-c)$, is orthogonal to $\underset{\sim}{n}$

i.e. $= 0$

$= 0$

this is often written as

$$n_1 x + n_2 y + n_3 z = d$$

where $d = n_1 a + n_2 b + n_3 c$

If $\underset{\sim}{n}$ is the normal to a surface $=$

this equation is

Chain rule for partial derivatives

Recall if $f : \mathbb{R} \to \mathbb{R}$, $u : \mathbb{R} \to \mathbb{R}$

then we can take the composition and the chain rule gives:

if we include arguments:

$$\frac{d}{dt} f \qquad = \frac{df}{du} \qquad \frac{du}{dt}$$

Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$, $v : \mathbb{R} \to \mathbb{R}$

then we can compose: $F(\quad , \quad)$

$\frac{dF}{dt}$ follows the chain rule for partial derivatives:

$$\frac{dF}{dt} =$$

<u>Example</u> $F(x,y) = x^2 + y^2$, $u(t) = t^2$, $v(t) = e^t$

find $\frac{d}{dt} F(u(t), v(t))$.

$\frac{du}{dt} =$, $\frac{dv}{dt} =$ $\qquad F(u,v) =$ (suppressing t for now)

$$\Rightarrow \frac{\partial F}{\partial u} = \qquad \frac{\partial F}{\partial v} =$$

so by the chain rule

$$\frac{dF}{dt}(u,v) =$$

expressed just in t: $\quad = \qquad =$

We can check this by substituting for t at the beginning and calculating $\frac{dF}{dt}$ directly:

$$F(u(t), v(t)) = \quad + \quad = \quad +$$

$$= \quad +$$

$$\frac{dF}{dt} = \quad +$$

Suppose now we have a vector-valued function $\underset{\sim}{g} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\underset{\sim}{g}(s,t) = (\quad , \quad)$$

– two coordinate functions!

since g maps into $\mathbb{R}^2$, and F takes its arguments from $\mathbb{R}^2$, they can be composed: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$F(\underset{\sim}{g}(s,t)) = F(g_1(s,t), g_2(s,t))$$

and we might need to find $\frac{\partial F}{\partial s}$ or $\frac{\partial F}{\partial t}$

$$\frac{\partial F}{\partial s} = \quad +$$

$$\frac{\partial F}{\partial t} = \quad +$$

In general if we have a multivariable function

$$f(u_1, u_2, \ldots, u_n)$$

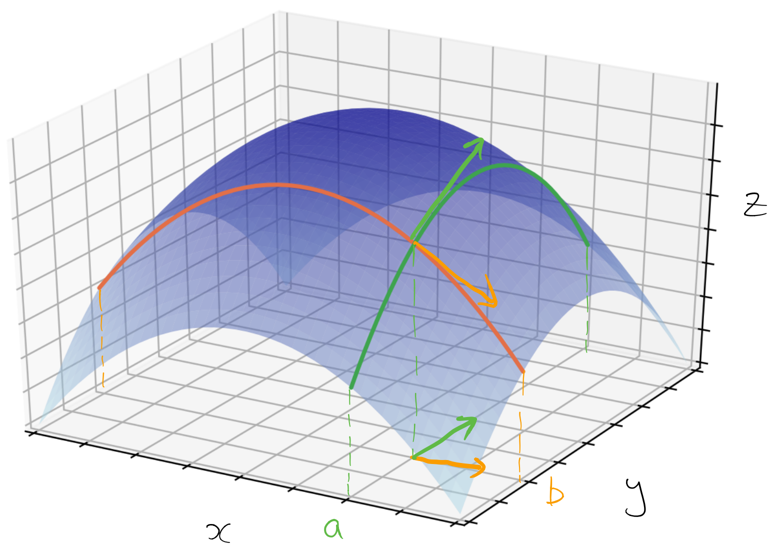
where each u_i is itself a multivariable function

$$u_i(t_1, t_2, \ldots, t_n)$$

then

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial t_i} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial u_n}\cdot\frac{\partial u_n}{\partial t_i}$$

Directional derivatives and differentiability



$z = f(x, y)$

$\dfrac{\partial f}{\partial x}$ rate of change of f (height) in x direction: $(1, 0)$

$\left(1, 0, \dfrac{\partial f}{\partial x}\right)$ tangent vector to the surface in x direction

$\dfrac{\partial f}{\partial y}$ rate of change of f (height) in y direction: $(0, 1)$

$\left(1, 0, \dfrac{\partial f}{\partial y}\right)$ tangent vector to the surface in x direction

Can we find the rate of change of f in some other direction?
(how steep is the ascent / descent in the direction $\underset{\sim}{v}$?)

The directional derivative of f at $\underset{\sim}{c} = (a, b)$ in the direction $\underset{\sim}{v}$ is defined by

$$D_{\underset{\sim}{v}} f(\underset{\sim}{c}) = \lim_{h \to 0} \frac{f(\underset{\sim}{c} + h\hat{\underset{\sim}{v}}) - f(\underset{\sim}{c})}{h} \qquad \text{where } \hat{\underset{\sim}{v}} =$$

if this limit exists then $D_{\underset{\sim}{v}} f(\underset{\sim}{c})$ can be expressed in terms of the partial derivatives of f:

$$D_{\underset{\sim}{v}} f(\underset{\sim}{c}) = \qquad\qquad \text{(proof below)}$$

Defining the gradient vector

$$\nabla f(a,b) = (\quad , \quad)$$

the directional derivative is therefore given by

$$D_{\underset{\sim}{v}} f(a,b) = \quad =$$

The same applies to directional derivatives of functions of more variables, eg: $\underset{\sim}{v} \in \mathbb{R}^3$, $f: \mathbb{R}^3 \to \mathbb{R}$

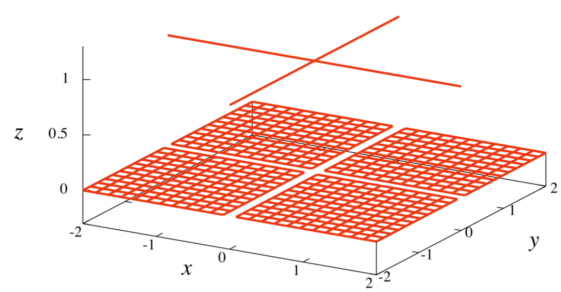
$$D_{\underset{\sim}{v}} f(a,b,c) = \nabla f(a,b,c) \cdot \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|}$$

where $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

A function $f: \mathbb{R}^2 \to \mathbb{R}$ is called differentiable at (a,b) if the directional derivative exists in every direction.

Note that the existence of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (a,b) doesn't guarantee that f is differentiable at (a,b),

eg: $$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$



$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

but no other directional derivatives exist. If we form a tangent plane using $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ at $(0,0)$ it won't be a good approximation!

It turns out that for differentiability at (a,b) we require $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to be continuous at (a,b) (and therefore they must exist for points around (a,b)).

Proof of the formula: $D_{\underset{\sim}{v}} f(\underset{\sim}{c}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \hat{\underset{\sim}{v}}$

First let $g(t) = f(\underset{\sim}{c} + t\hat{\underset{\sim}{v}})$, then from the definition of $\frac{dg}{dt}$:

$$\begin{aligned}
\frac{dg}{dt}(0) &= \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} \\
&= \lim_{h \to 0} \frac{f(\underset{\sim}{c} + h\hat{\underset{\sim}{v}}) - f(\underset{\sim}{c})}{h} \qquad \leftarrow \text{since } \begin{array}{l} g(h) = f(\underset{\sim}{c} + h\hat{\underset{\sim}{v}}) \\ g(0) = f(\underset{\sim}{c}) \end{array} \\
&= D_{\underset{\sim}{v}} f(\underset{\sim}{c}) \qquad \text{(by definition)}
\end{aligned}$$

this proves

$$D_{\underset{\sim}{v}} f(\underset{\sim}{c}) = \left.\frac{d}{dt}\right|_{t=0} f(\underset{\sim}{c} + t\hat{\underset{\sim}{v}})$$

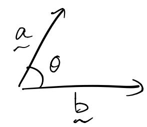
to which we will apply the chain rule:

let $\underset{\sim}{c} + t\hat{\underset{\sim}{v}} = \begin{pmatrix} a \\ b \end{pmatrix} + t \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix} = \begin{pmatrix} a + t\hat{v}_1 \\ b + t\hat{v}_2 \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$\begin{aligned}
\frac{d}{dt} f(\underset{\sim}{c} + t\hat{\underset{\sim}{v}}) &= \frac{d}{dt} f(a + t\hat{v}_1, b + t\hat{v}_2) \\
&= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \\
&= \frac{\partial f}{\partial x} \cdot \hat{v}_1 + \frac{\partial f}{\partial y} \cdot \hat{v}_2 \\
&= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \hat{\underset{\sim}{v}}
\end{aligned}$$

Maximum rate of change

Directional derivatives give us a way of calculating the rate of change in <u>any</u> direction, so we might ask which direction gives the maximum or minimum rate of change. (at a given point on a mountain, what is the direction of steepest ascent / descent?)

Recall: $\underset{\sim}{a} \cdot \underset{\sim}{b} =$ where 

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, then

$$D_{\underset{\sim}{v}} f(\underset{\sim}{c}) = \qquad\qquad =$$

but $\hat{\underset{\sim}{v}} = \dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$ so $|\hat{\underset{\sim}{v}}| =$, $|\nabla f(\underset{\sim}{c})|$ doesn't depend on $\underset{\sim}{v}$,

"unit vector"

so the maximum of $D_{\underset{\sim}{v}} f(\underset{\sim}{c})$ (for $\underset{\sim}{c}$ fixed) occurs at the maximum of $\cos\theta$, which is 1 at $\theta = 0$.

$\theta = 0$ means $\nabla f(\underset{\sim}{c})$ and $\hat{\underset{\sim}{v}}$ are in the same direction, so $\hat{\underset{\sim}{v}}$ is a unit vector in the $\nabla f(\underset{\sim}{c})$ direction, i.e. $\hat{\underset{\sim}{v}} =$

Therefore:

The maximum of $D_{\underset{\sim}{v}} f(\underset{\sim}{c})$ occurs when $\hat{\underset{\sim}{v}} = \dfrac{\nabla f(\underset{\sim}{c})}{|\nabla f(\underset{\sim}{c})|}$

Similarly, the minimum occurs at min $\cos\theta$, i.e. $\theta = -\pi$ which means $\hat{\underset{\sim}{v}}$ (and therefore $\underset{\sim}{v}$) is in the opposite direction to $\nabla f(\underset{\sim}{c})$

The minimum of $D_{\underset{\sim}{v}} f(\underset{\sim}{c})$ occurs when $\hat{\underset{\sim}{v}} = -\dfrac{\nabla f(\underset{\sim}{c})}{|\nabla f(\underset{\sim}{c})|}$

Example 3.49. *The temperature at each point of a metal plate is given by the function $T(x,y) = e^x \cos y + e^y \cos x$. In what direction does the temperature increase most rapidly at the point $(0,0)$. What is this rate of increase?*

The direction of maximum rate of change of temperature is the direction of $\nabla T(0,0)$

$\nabla T(x,y) = \qquad\qquad = (\qquad\qquad)$

So

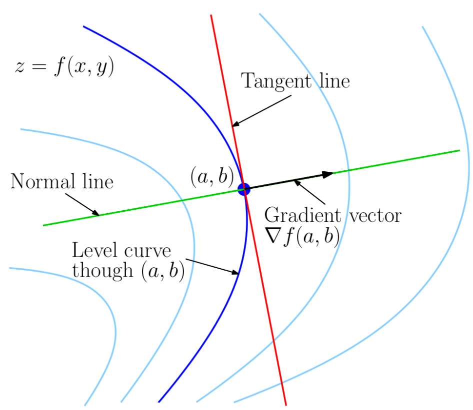
$\nabla T(0,0) =$

The rate of change of T at $(0,0)$ in this direction is

$D_{\nabla T(0,0)} T(0,0) = \qquad = \qquad =$

$=$

Contrast with the level curves, ,curves of equal height. So a tangent vector to a level curve gives a direction of no change in height. These directions turn out to be perpendicular to the directions of max/min rate of change! (steepest ascent/descent)



A similar result holds for level surfaces $f(x,y,z) = k$ $(f: \mathbb{R}^3 \to \mathbb{R})$

The gradient vector $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ at a point (a,b,c) on the level surface $f(x,y,z) = k$ is perpendicular to every tangent vector to the level surface at (a,b,c), i.e. perpendicular to the tangent plane, i.e. a normal vector to the surface!

The Jacobian matrix

$f : \mathbb{R}^2 \to \mathbb{R}$ has a gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

which gives rates of change of f in different directions

If $\underset{\sim}{h} : \mathbb{R}^2 \to \mathbb{R}^2$, then it has two coordinate functions

$$\underset{\sim}{h}(x,y) = (\qquad\qquad\qquad)$$

$\underset{\sim}{h}$ is a vector quantity changes in the value of $\underset{\sim}{h}$ can happen in two directions, so the change in $\underset{\sim}{h}$ in the x-direction (or y direction etc.) is a vector quantity made up of the change in h_1 and the change in h_2.

The Jacobian matrix, also called the (total) derivative of $\underset{\sim}{h}$, is

$$D\underset{\sim}{h} = \begin{bmatrix} \qquad\qquad \end{bmatrix}$$

There is a lot more to be said about this derivative (extension to functions $\mathbb{R}^2 \to \mathbb{R}^3$, $\mathbb{R}^m \to \mathbb{R}^n$, matrix chain rule ...) but for this course the important thing is the Jacobian which is the determinant of the Jacobian matrix $\det(D\underset{\sim}{h})$

Example

$\underset{\sim}{h}(x,y) = (x^2, x+y)$ find the Jacobian of h

$$\frac{\partial h_1}{\partial x} = 2x \qquad \frac{\partial h_1}{\partial y} = 2y$$