Extrema of functions of one variable

$D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$

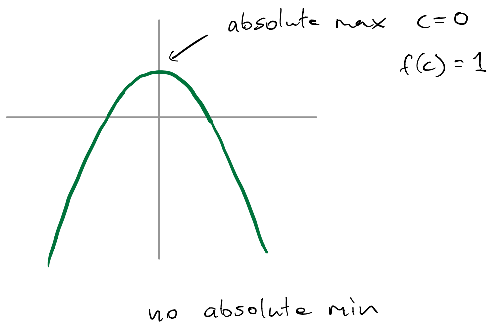
f has an absolute maximum at $c \in D$ if $f(x) \leq f(c)$ for all $x \in D$

f has an absolute minimum at $c \in D$ if $f(x) \geq f(c)$ for all $x \in D$

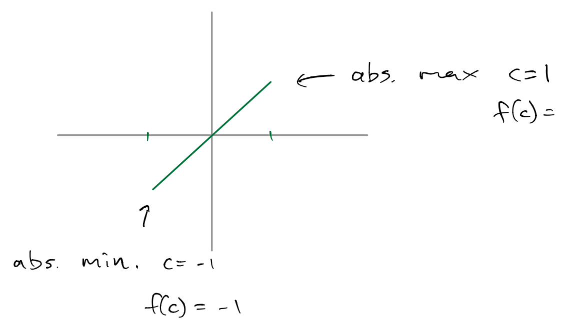
the domain matters!

Examples

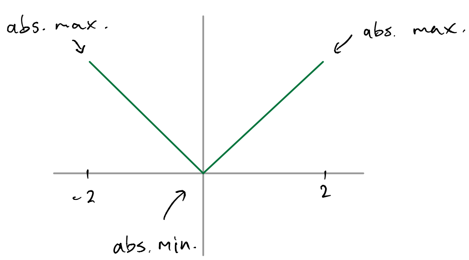
$f : \mathbb{R} \to \mathbb{R}$
$f(x) = 1 - x^2$



$g : [-1,1] \to \mathbb{R}$, $g(x) = x$



$f : [-2,2] \to \mathbb{R}$, $f(x) = |x|$

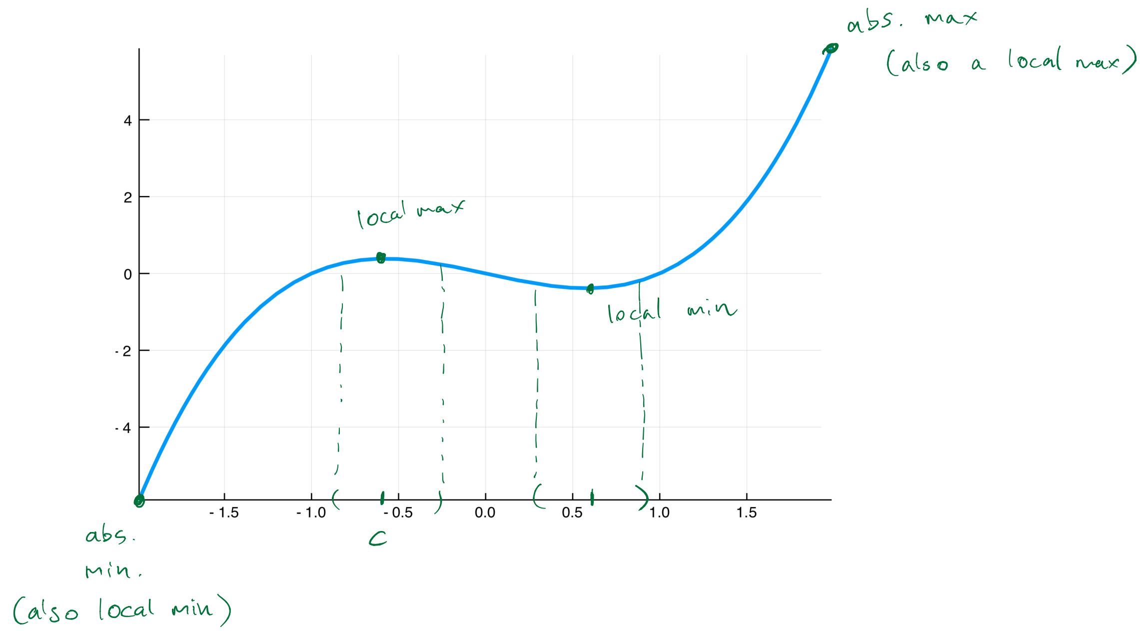


f has a local maximum at $c \in D$ if there is an open interval (a,b) containing c such that $f(x) \leq f(c)$ for all $x \in (a,b) \cap D$.

f has a local minimum at $c \in D$ if there is an open interval (a,b) containing c such that $f(x) \geq f(c)$ for all $x \in (a,b) \cap D$.

Examples

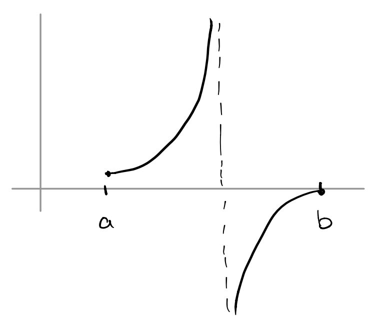
$$f : [-2, 2] \to \mathbb{R}, \qquad f(x) = (x-1)^2 (x+1)$$



Extreme value theorem

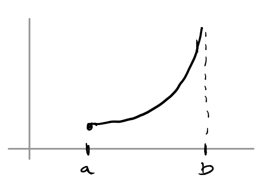
Let $f : [a, b] \to \mathbb{R}$ be a continuous function, then f has an absolute maximum and an absolute minimum.

Note: If f is not continuous the theorem is not true eg:



If the domain is an open-ended interval $[a, b)$, (a, b), $(a, b]$ then f can be continuous and still have no abs. max/min

eg: $[a, b)$



Boundary points of domains $D \subseteq \mathbb{R}$.

$D = [1,2)$ has boundary point 1

$D = [a,b]$ has boundary points $a, b \in \mathbb{R}$

$D = (-\infty,-1) \cup (-1,\infty)$ has no boundary points

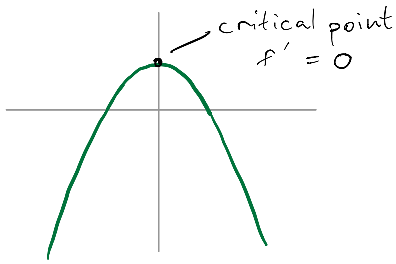
$D = [-5,-1] \cup [0,1) \cup (1,5]$ has boundary points $-5, -1, 0, 5$

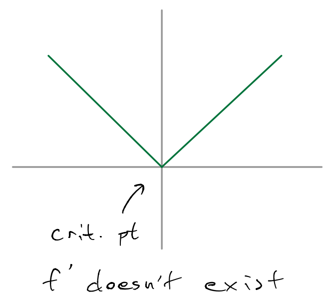
PRECISE DEFINITION

Let $D \subset \mathbb{R}$, $c \in \mathbb{R}$ is called a boundary point of D if every open interval (a,b) containing c contains points that are not in D as well as points in D.

$c \in D$ is called an interior point if it is not a boundary point.

If $f: D \to \mathbb{R}$, $c \in D$ is called a critical point of f if it is an interior point of D and either $f'(c) = 0$ or $f'(c)$ doesn't exist



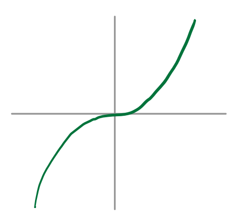


FERMAT'S THEOREM

If $f: D \to \mathbb{R}$ has a local max or min at $c \in D$, and c is an interior point of D, then c is a critical point of f.

Note: the converse is not true: a critical point is not necessarily a local extremum. eg: $f(x) = x^3$, $f'(x) = 2x^2$, $f'(0) = 0$

but $x = 0$ is a point of inflection:



PUTTING THE TWO THEOREMS TOGETHER:

- If $f : [a,b] \to \mathbb{R}$ is continuous then it has absolute extrema (by the extreme value theorem).
- The extrema are either boundary points or interior points (there are no other points)
- If an interior point is to be a candidate for an absolute extremum it must at least be a local extremum, in which case it is a critical point (by Fermat's theorem).

Therefore, to find the absolute extrema of a continuous function $f : [a,b] \to \mathbb{R}$

- Find $f(a)$ and $f(b)$
- Find the critical points $c_1, c_2, \ldots \in (a,b)$ and compare the values $f(a), f(b), f(c_1), f(c_2), \ldots$ to find absolute max/min.

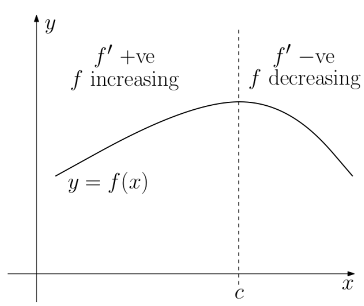
Identifying local maxima and minima

Suppose c is a critical point of $f: D \subset \mathbb{R} \to \mathbb{R}$. How can we check if c is a local max, local min or neither?

We look at two methods.

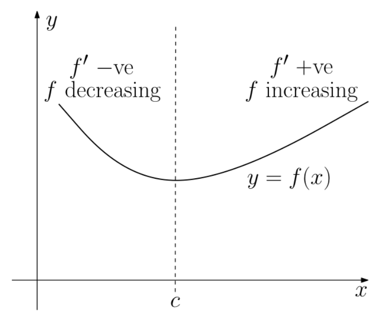
First derivative test

Check f' either side of c



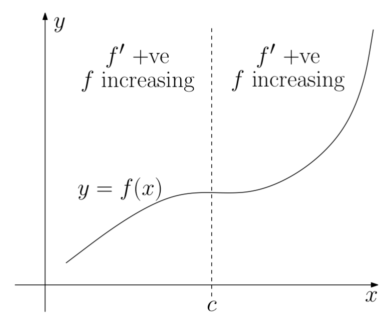
concave down

local max



concave up

local min

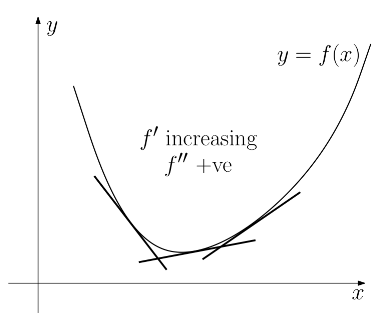


neither

Second derivative test

Find $f''(c)$

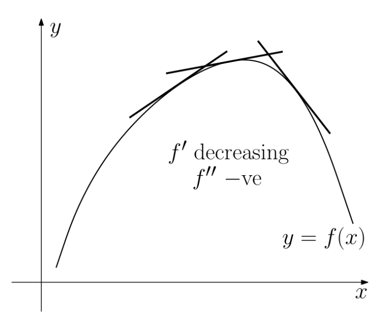
$f''(c) > 0$



concave up

local min

$f''(c) < 0$



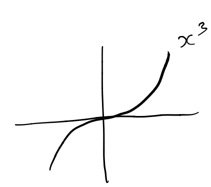
concave down

local max

If $f''(0) = 0$ the test is inconclusive

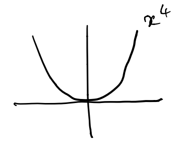
eg: $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$

$f'(0) = 0$ $\quad f''(0) = 0$



but $\quad f(x) = x^4, \quad f'(x) = 4x^3, \quad f''(x) = 12x^2$

$$f'(0) = 0, \quad f''(0) = 0$$



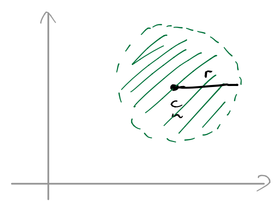
Extrema of functions of two variables

Domains in $\mathbb{R}^2$

An open disc (or ball) centred at $\underset{\sim}{c} \in \mathbb{R}^2$ is any set of the form

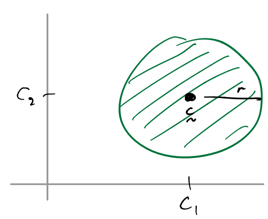
$$\{ (x,y) \in \mathbb{R}^2 : (x-c_1)^2 + (y-c_2)^2 < r^2 \}$$

r is the radius.



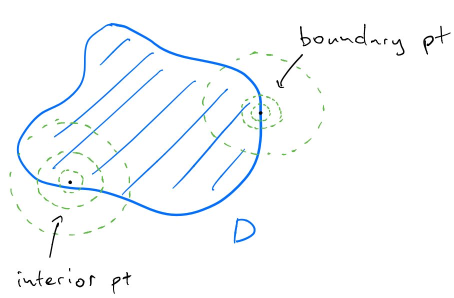
An closed disc (or ball) centred at $\underset{\sim}{c} \in \mathbb{R}^2$ is any set of the form

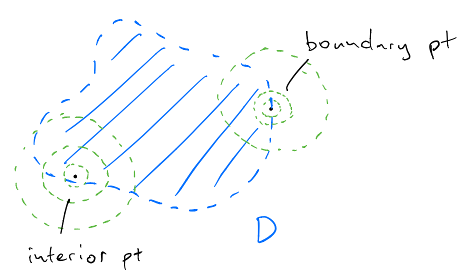
$$\{ (x,y) \in \mathbb{R}^2 : (x-c_1)^2 + (y-c_2)^2 \leq r^2 \}$$



A closed disc contains its boundary, an open disc does not.

For a general domain D boundary points can be characterised as follows: $\underset{\sim}{x} \in \mathbb{R}^2$ is a boundary point of D if every open disc centred at $\underset{\sim}{x}$ contains points outside D.





∂D is the set of all boundary points of D

closed

D $\qquad$ ∂D

neither open nor closed

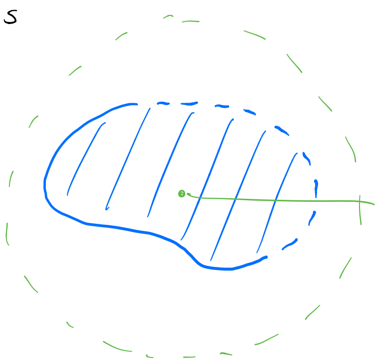
open

D is called closed if it contains its boundary: $\partial D \subset D$

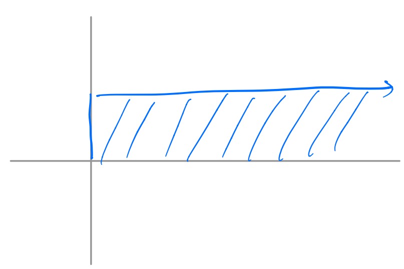
D is called open if it doesn't contain any boundary points:

$$\partial D \cap D = \emptyset$$

D is called bounded if it is contained in a disc of finite radius



bounded



not bounded

Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$. We say f has an absolute maximum at $\underset{\sim}{c} \in D$ if $f(\underset{\sim}{x}) \leq f(\underset{\sim}{c})$ for all $x \in D$.

(abs. min $f(\underset{\sim}{x}) \geq f(\underset{\sim}{c})$)

We say f has a local maximum at $\underset{\sim}{c}$ if there exists an open disc B centred at $\underset{\sim}{c}$ such that $f(\underset{\sim}{x}) \leq f(\underset{\sim}{c})$ for all $x \in B$.

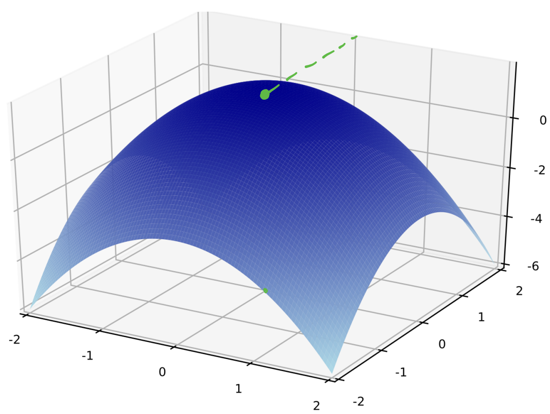
(local min. $f(\underset{\sim}{x}) \geq f(\underset{\sim}{c})$)

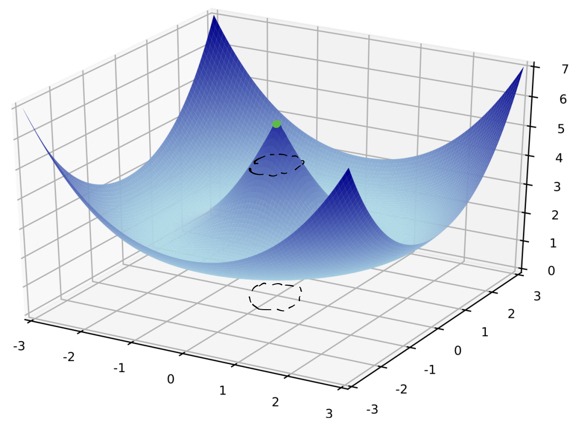
$f(x,y) = 2 - (x^2+y^2)$

absolute max at $(0,0)$

$f(x,y) = 2 + (x^2+y^2-2)^2$

local max at $(0,0)$





EXTREME VALUE THEOREM

Let D be a non-empty closed and bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ a continuous function. Then f has an absolute maximum and an absolute minimum.

Critical points of functions of 2 variables

Let $D \subset \mathbb{R}^2$ and $f: D \to \mathbb{R}$ a function.

Recall: If f is differentiable at a point $\underset{\sim}{c} \in D$, the gradient $\nabla f(\underset{\sim}{c})$ gives the direction of greatest increase of f. The negative gradient $-\nabla f(\underset{\sim}{c})$ gives the direction of most rapid decrease.

At a local maximum $f(\underset{\sim}{c})$:

- moving in any direction will lead to a decrease in the value of f
- since there is no direction of increase, if the gradient vector exists it must be zero $\nabla f(\underset{\sim}{c}) = \underset{\sim}{0}$

At a local minimum $f(\underset{\sim}{c})$:

- moving in any direction will lead to an increase in the value of f
- since there is no direction of decrease, if the gradient vector exists it must be zero $\nabla f(\underset{\sim}{c}) = \underset{\sim}{0}$

We say $\underset{\sim}{c} \in D$ is a critical point of f if

- $\underset{\sim}{c}$ is an interior point of D, and
- $\nabla f(\underset{\sim}{c})$ doesn't exist or $\nabla f(\underset{\sim}{c}) = \underset{\sim}{0}$

Examples

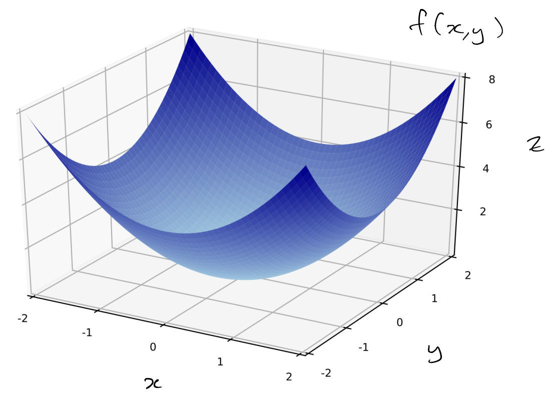
$f(x,y) = x^2 + y^2$, $\quad g(x,y) = 1 - x^2 - y^2$, $\quad h(x,y) = y^2 - x^2$,

each of these has a critical point at $(0,0)$:

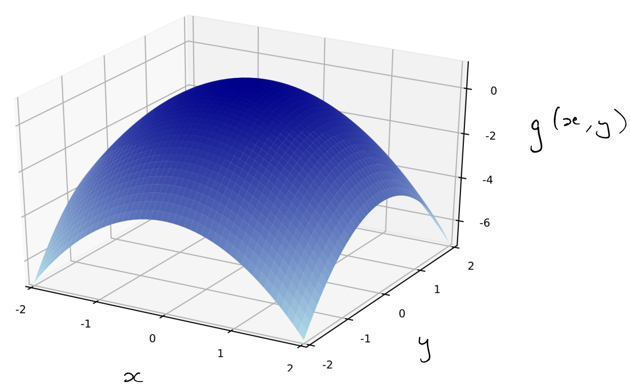
$\nabla f(x,y) = (2x, 2y)$ $\quad \nabla g(x,y) = (-2x, -2y)$ $\quad \nabla h(x,y) = (-2x, 2y)$

$\nabla f(0,0) = (0,0)$ $\quad \nabla g(0,0) = (0,0)$ $\quad \nabla h(x,y) = (0,0)$

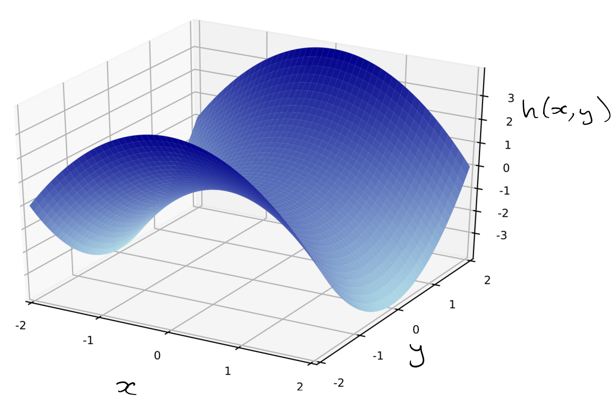
Since $f(x,y) = x^2 + y^2 \geq 0$, $f(0,0) = 0$ is an absolute minimum



$g(x,y) = 1 - x^2 - y^2 \leq 1$, $g(0,0) = 1$ is an absolute max



At the critical point $(0,0)$ $h(x,y)$ decreases in one direction but increases in another, $(0,0)$ is called a saddle point



The second derivative test

$D \subset \mathbb{R}^2$, $\quad f : D \to \mathbb{R}$

If all second-order partial derivatives of f exist at $\underset{\sim}{c} \in D$, we define the Hessian matrix at $\underset{\sim}{c}$:

$$D^2 f(\underset{\sim}{c}) = \begin{bmatrix} f_{xx}(\underset{\sim}{c}) & f_{xy}(\underset{\sim}{c}) \\ f_{yx}(\underset{\sim}{c}) & f_{yy}(\underset{\sim}{c}) \end{bmatrix}$$

Denote $\quad D_{\underset{\sim}{c}} = \det\left(D^2 f(\underset{\sim}{c})\right)$

If $\nabla f(\underset{\sim}{c}) = \underset{\sim}{0}$ and all the second partial derivatives of f exist and are continuous on an open disc containing $\underset{\sim}{c}$ then

- if $D_c > 0$ and $f_{xx}(\underset{\sim}{c}) > 0$ then f has a local min. at $\underset{\sim}{c}$
- if $D_c > 0$ and $f_{xx}(\underset{\sim}{c}) < 0$ then f has a local max at $\underset{\sim}{c}$
- if $D_c < 0$ then $\underset{\sim}{c}$ is a saddle point of f
- if $D_c = 0$ the second derivative test is inconclusive.

Example $\qquad f : \mathbb{R}^2 \to \mathbb{R}$

Find all critical points of $f(x,y) = x^3 - 3xy + y^3$ and determine their type (if possible).

$\nabla f(x,y) = (3x^2 - 3y,\ 3y^2 - 3x) \qquad$ exists everywhere

$= (0, 0)$

two equations: $\quad 3x^2 - 3y = 0 \quad \to \quad y = x^2$

$3y^2 - 3x = 0 \quad \to \quad 3x^4 - 3x = 0$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or $x^3 = 1$

$x = 1$

$y = 0$ or $y = 1$

the critical points are $(0,0)$ and $(1,1)$

The Hessian $\quad f_{xx} = 6x \quad f_{xy} = -3 = f_{yx} \quad f_{yy} = 6y \quad \Big\}$ all continuous

$$D^2 f = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

at $(0,0)$

$$D^2 f(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \qquad D_{(0,0)} = -9 < 0$$

saddle point

at $(1,1)$

$$D^2 f(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \qquad D_{(1,1)} = 36 - 9 = 27 > 0$$

$f_{xx} = 6 > 0$ local min.