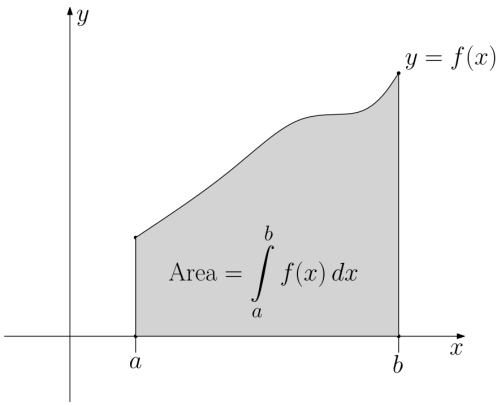
Integration



$f : [a,b] \to \mathbb{R}$

$$\int_a^b f(x)\,dx$$

$f(x)$ integrand

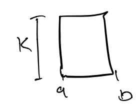
a, b limits of integration

$\int_a^b$ definite integral

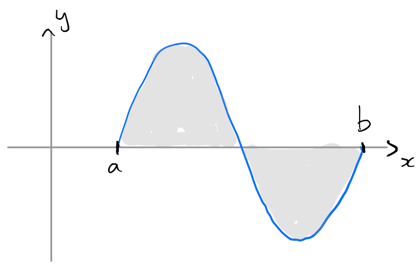
- output is a number.

When f is continuous on $[a,b]$ this area is finite by the extreme value theorem: if K is the absolute maximum of f then

$$\int_a^b f(x)\,dx \leq K\,[a,b] \quad \leftarrow \text{area of the rectangle:}$$



Signed area



Areas below the x-axis count toward the integral with a minus sign

$\leftarrow \int_a^b f(x)\,dx = 0$, the two areas cancel.

A function $F(x)$ is called an anti-derivative for $f(x)$ if

$$F'(x) = f(x)$$

FUNDAMENTAL THEOREM OF CALCULUS

If F is an anti-derivative for f then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

notation: $= F(x)\Big|_a^b = \big[F(x)\big]_a^b$

The indefinite integral $\int f(x)\,dx$ is all the antiderivatives

i.e. if $F(x)$ is any antiderivative for f then $\int f(x)\,dx = F(x) + C$

$C \in \mathbb{R}$

definite integral $\rightarrow$ signed area $\in \mathbb{R}$

indefinite integral $\rightarrow$ family of functions $(+C)$

A table of things you already know

$\frac{dy}{dx}$	y	$\int y\,dx$
0	a (constant)	$ax + C$
nx^{n-1}	x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
$-\frac{1}{x^2}$ or $-x^{-2}$	$\frac{1}{x}$ or x^{-1}	$\ln x + C$
e^x	e^x	$e^x + C$
$\frac{1}{x}$	$\ln x$	
$\cos x$	$\sin x$	$-\cos x + C$
$-\sin x$	$\cos x$	$\sin x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	
$-\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	
$\frac{1}{1+x^2}$	$\tan^{-1} x$	

Integration techniques

Substitution: undoing the chain rule:

if $F(x)$ is an antiderivative for $f(x)$, then $F'(x) = f(x)$, so

chain rule: $$\frac{d}{dx} F(u(x)) = F'(u(x))\, u'(x) = f(u(x))\, u'(x)$$

i.e. $$f(u(x))\, u'(x) = \frac{d}{dx} F(u(x))$$

now integrating each side: $\int \frac{d}{dx} F(u(x))\, dx = F(u(x)) + C$
$= \int f(u)\, du$

indefinite $$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

definite $$\int_{x=a}^{x=b} f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

Example $$\int \sqrt{x^2 + 4x + 7}\, (x+2)\, dx$$

Let $u = x^2 + 4x + 7$ so $\frac{du}{dx} = (2x+4)$

$$dx = \frac{1}{2x+4}\, du$$

substitute u and dx:

$$\int \sqrt{x^2 + 4x + 7}\, (x+2)\, dx = \int \sqrt{u}\, (x+2)\, \frac{1}{2x+4}\, du$$

$$= \int \sqrt{u}\, \frac{x+2}{2(x+2)}\, du$$

$$= \int \frac{\sqrt{u}}{2}\, du$$

$$= \frac{u^{3/2}}{3} + C = \frac{1}{3}(x^2 + 4x + 7)^{3/2} + C$$

Example $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x \, dx$

let $u = \sin x$ $\qquad \dfrac{du}{dx} = \cos x \, , \quad dx = \dfrac{1}{\cos x} du$

limits $\quad x = 0 : \quad u(0) = 0$

$\qquad\quad x = \frac{\pi}{2} : \quad u(\frac{\pi}{2}) = 1$

subst. $\quad u, \; dx, \;$ limits

$$\int_0^{\pi/2} \sin^2 x \cos x \, dx = \int_0^1 u^2 \cos x \frac{1}{\cos x} du$$

$$= \int_0^1 u^2 du = \left[\frac{u^3}{3} \right]_0^1 = \frac{1}{3}$$

Using trig identities

how to integrate $\int \sin^2 x \, dx$ or $\int \cos^2 x \, dx$.

identities : $\quad \sin^2 x + \cos^2 x = 1 \quad \rightarrow \quad \sin^2 x = 1 - \cos^2 x$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cos^2 x = 1 - \sin^2 x$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\left.\begin{aligned} &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}\right\} \text{ rearrange: } \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

so eg: $\quad \displaystyle\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx$

$$= \frac{x}{2} - \frac{\sin 2x}{4}$$

In general: integrating $\int \sin^m x \cos^n x \, dx$ requires some combination of the above identities with a substitution.

Other identities: eg: $\int \sin(4x)\cos(5x)\,dx$

use the product formula: (derived from addition formula)

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\sin(4x)\cos(5x) = \frac{1}{2}\left(\sin 9x + \sin(-x)\right) = \frac{1}{2}\sin 9x - \frac{1}{2}\sin x$$

now integrate...

Inverse trig substitutions

Example integrate $\int \frac{2x}{\sqrt{4-x^2}}\,dx$

$u = 4 - x^2 \qquad \frac{du}{dx} = -2x \qquad dx = \frac{-1}{2x}\,du$

subst.

$$\int \frac{2x}{\sqrt{u}} \cdot \left(\frac{-1}{2x}\right) du = \int \frac{-1}{\sqrt{u}}\,du = \int -u^{-\frac{1}{2}}\,du$$

$$= -2u^{\frac{1}{2}} = -2\sqrt{u}$$

$$= -2\sqrt{4-x^2}$$

what about $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$?

the substitution $u = 4 - x^2$ doesn't work with x^2 in the numerator. ($\frac{du}{dx}$ doesn't cancel out)

solution: substitute $u = \sin^{-1}\left(\frac{x}{2}\right)$ (not obvious)

for this substitution it is easier to calculate $\frac{dx}{du}$ instead of $\frac{du}{dx}$

so from $u = \sin^{-1}\left(\frac{x}{2}\right)$ take sin of each side:

$$\sin u = \frac{x}{2}$$

$$x = 2\sin u$$

$$\frac{dx}{du} = 2\cos u \quad \Rightarrow \quad dx = 2\cos u\,du$$

now subst. x and dx

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2 u}{\sqrt{4 - 4\sin^2 u}}\,2\cos u\,du$$

$$= \int \frac{4\sin^2 u}{\sqrt{4(1-\sin^2 u)}}\,2\cos u\,du$$

$$= \int \frac{4 \sin^2 u}{2\sqrt{\cos^2 u}} \, 2 \cos u \, du$$

$$= \int 4 \sin^2 u \, du$$

$$= \int \frac{4}{2}(1 - \cos 2u) \, du$$

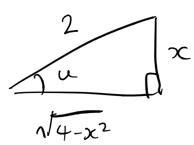
$$= 2u - \sin 2u + C$$

$$= 2u - 2 \sin u \cos u + C$$

to get this in terms of x (neatly):

$$x = 2 \sin u$$

$$\sin u = \frac{x}{2}$$



$$\longrightarrow \quad \cos u = \frac{\sqrt{4-x^2}}{2}$$

so

$$2u - 2 \sin u \cos u = 2 \sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\frac{\sqrt{4-x^2}}{2}$$

Partial fractions

Example $f(x) = \dfrac{3x-1}{x^2-1}$ is not easy to integrate (try!)

however, observe that

$$\frac{2}{x+1} + \frac{1}{x-1} \underset{\text{common denominator}}{=} \frac{2(x-1) + (x+1)}{(x+1)(x-1)} = \frac{3x-1}{x^2-1}$$

$$\int \text{LHS}\, dx = 2\ln|x+1| + \ln|x-1| + C \qquad \leftarrow \text{easy (ish)}$$

So how do we reverse the common denominator technique?

Partial fractions

This applies to functions of the form $\dfrac{P(x)}{Q(x)}$ where $P(x)$, $Q(x)$ are polynomials and the degree (highest power) of P is less than the degree of Q:

Case 1: Denominator has distinct linear factors

$$f(x) = \frac{P(x)}{(x-a_1)\cdots(x-a_k)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_k}{x-a_k},$$

where $a_1, \ldots, a_k$ are pairwise distinct.

Case 2: Denominator has repeated linear factors

$$f(x) = \frac{P(x)}{(x-a)^c} = \frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \cdots + \frac{B_{c-1}}{(x-a)^{c-1}} + \frac{B_c}{(x-a)^c}.$$

Case 3: Denominator has an irreducible factor of degree 2

$$f(x) = \frac{P(x)}{(x-a)(x^2+bx+c)} = \frac{A_1}{x-a} + \frac{C_1x + C_2}{x^2+bx+c}.$$

irreducible means it can't be factorised - the quadratic formula will give complex solutions.

Example $\dfrac{3x-1}{x^2-1} = \dfrac{3x-1}{(x-1)(x+1)}$ factorise denominator

$$= \frac{A}{x-1} + \frac{B}{x+1} \qquad (\text{Case } 1)$$

now multiply each side by the factorised denominator

$$3x - 1 = \frac{A}{x-1}(x-1)(x+1) + \frac{B}{x+1}(x-1)(x+1)$$

$$3x - 1 = A(x+1) + B(x-1)$$

At this point there are two ways to proceed, use whichever you prefer

① substitute $x = -1$ (so that $(x+1) = 0$)

$$3(-1) - 1 = 0 + B(-2)$$

$$-4 = -2B$$

$$B = 2$$

substitute $x = 1$ (so that $(x-1) = 0$)

$$3(1) - 1 = A(2) + 0$$

$$2 = 2A$$

$$A = 1$$

Therefore

$$\frac{3x-1}{x^2-1} = \frac{1}{x-1} + \frac{2}{x+1}$$

② $$3x - 1 = A(x+1) + B(x-1)$$

$$3x - 1 = Ax + A + Bx - B$$

equate constants $(x=0)$

$$-1 = A - B$$

equate coefficients of x

$$3 = A + B$$

solve simultaneously:

adding eqns: $2 = 2A$

$A = 1$

$B = 2$

More examples in the unit reader

Integration by parts

Undoing the product rule

Product rule: $\frac{d}{dx}\left(u(x)\,v(x)\right) = u\,v' + u'\,v$

integrate each side wrt x,

indefinite $\quad u(x)\,v(x) = \int u\,v'\,dx + \int u'\,v\,dx$

definite $\quad \left[u(x)\,v(x)\right]_a^b = \int_a^b u\,v'\,dx + \int_a^b u'\,v\,dx$

These formulas are actually most useful after rearranging:

Integration by parts formula: $\quad u(x),\ v(x)$

indefinite form $\quad \int u\,v'\,dx = u\,v - \int u'\,v\,dx$

definite $\quad \int_a^b u\,v'\,dx = \left[u\,v\right]_a^b - \int_a^b u'\,v\,dx$

How to use:

$$\int \underbrace{x}_{u}\,\underbrace{\sin x}_{v'}\,dx = \underbrace{x}_{u}\,\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\,dx$$

$v = -\cos x \qquad u' = 1$

$$= -x\cos x + \sin x + C$$

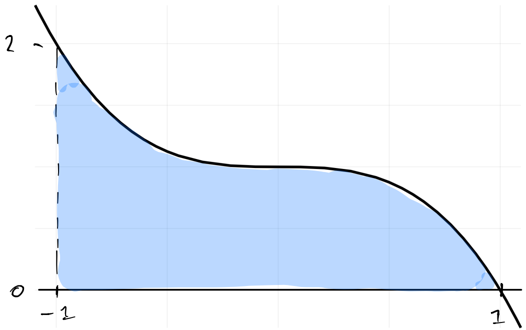
$$\int \underbrace{x}_{v'}\,\underbrace{\ln x}_{u}\,dx = \underbrace{(\ln x)}_{u}\,\underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{1}{x}}_{u'}\cdot\underbrace{\frac{x^2}{2}}_{v}\,dx$$

$v = \frac{x^2}{2}$

$$= (\ln x)\,\frac{x^2}{2} - \frac{x^2}{4} + C$$

Plan ahead when assigning u and v' so that v is easy to find and $\int u'\,v\,dx$ is easier than the original integral.

Riemann sums and integrals



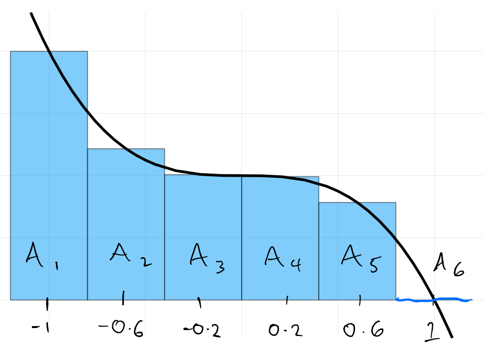
$f(x) = 1 - x^3$

$$\int_{-1}^{1} 1 - x^3 = \underbrace{\left[x - \frac{x^4}{4} \right]_{-1}^{1}}_{\text{antiderivative}} = 2$$

SOME FUNCTIONS DON'T HAVE ANTIDERIVATIVES!

For example e^{-x^2}. So how do we compute eg: $\int_{-1}^{1} e^{-x^2}$?

Back to basics:



$\Delta x = 0.4$

$A_1 = f(-1)\,\Delta x = 2 \times 0.4 = 0.8$

$A_2 = f(-0.6)\,\Delta x \approx 0.49$

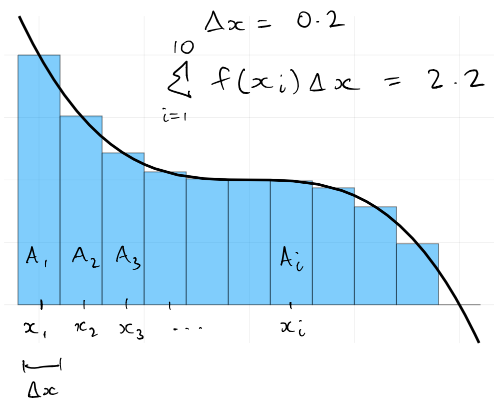
$A_3 \approx 0.4 \qquad A_4 \approx 0.4 \qquad A_5 \approx 0.3$

Total area $= 2.4$

Formula: $A_i = f(x_i)\,\Delta x$ Total Area $= \sum_i f(x_i)\,\Delta x$

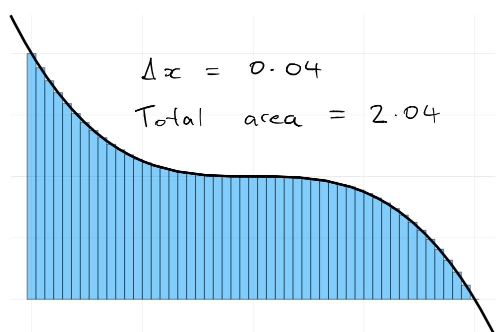
$\Delta x = 0.2$

$$\sum_{i=1}^{10} f(x_i)\,\Delta x = 2.2$$



$\Delta x = 0.04$

Total area $= 2.04$



Riemann sum $\sum_{i=1}^{n} f(x_i)\,\Delta x$ (total area of the bars)

Riemann integral $\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x \to 0}} \sum_i^n f(x_i)\,\Delta x$

if the limit exists f is called Riemann integrable over $[a,b]$

Notes

Here we have used (midpoint of the interval) intervals of the form $[x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}]$ and we evaluate $f(x_i)$ in the Riemann sum. There are several alternatives which turn out to give the same result:

- intervals $[x_i, x_i + \Delta x]$
- intervals of varying size: $[x_i, x_i + \Delta x_i]$
- $\sum f(c_i)\,\Delta x_i$ where c_i is any element in the i^{th} interval.

It is possible to prove (see unit reader for a sketch-proof) that the Riemann integral satisfies the Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$

So all the integration you have done so far (i.e. using anti-derivatives) has been Riemann integration.

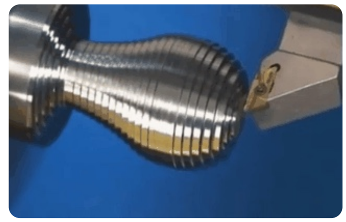
It is also possible to prove that if f is bounded and piecewise continuous (i.e. has only finitely many points of discontinuity) then it is Riemann integrable.

So alternative definitions of integration (eg: Lebesgue integration) are only needed when considering weird functions

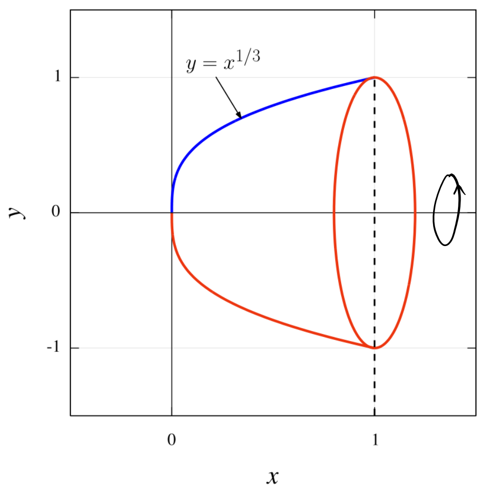
Applications of Riemann Sums.

Thinking about integrals in terms of Riemann sums gives us a good way of figuring out how to solve real problems using integrals (as opposed to just finding areas under curves). As usual, this is best demonstrated with examples.

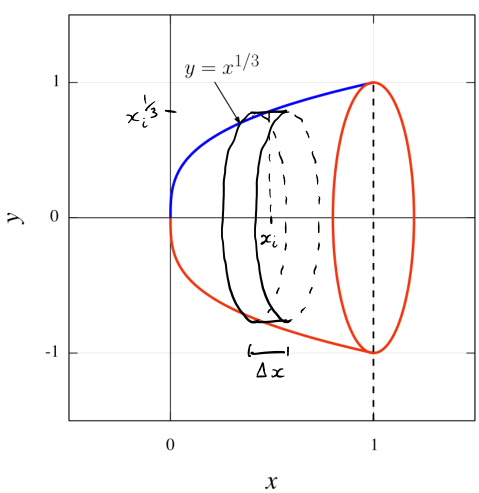
Volumes by cross-sections (rotational solids)



Suppose a piece of metal is cut using a lathe along the curve $y = x^{1/3}$, $0 < x < 1$ so that the remaining object has the shape of the solid obtained by rotating the area below $y = x^{1/3}$ about the x-axis:



we can find the volume of this object by setting up a Riemann integral.



The cylinder of width Δx at x_i has radius $y = x_i^{1/3}$ and therefore volume $\pi (x_i^{1/3})^2 \Delta x$

A Riemann sum of n such cylinders:

$$V \approx \sum_{i=1}^{n} \pi x_i^{2/3} \Delta x$$

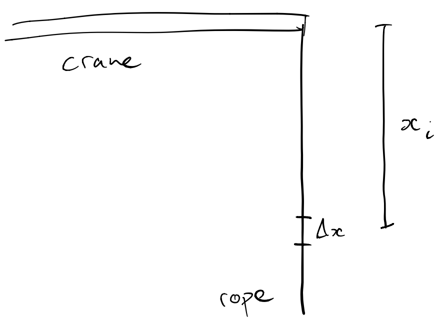
In the limit $n \to \infty$ and $\Delta x \to 0$

$$\sum_{i=1}^{n} \pi x_i^{2/3} \Delta x \longrightarrow \int_0^1 \pi x^{2/3}\, dx = \left[\frac{3\pi x^{5/3}}{5} \right]_0^1 = \frac{3\pi}{5}$$

So the volume of the remaining solid is $\frac{3\pi}{5}$

Work done by a force

Consider a rope with uniform density 0.1 kg/m and length 50m which is hanging from a crane. Suppose we need to calculate how much work it will take to lift all the rope to the top of the crane. Remember work = force × displacement



Consider a small section of rope with length Δx which is x_i metres from the top. The mass of this piece is $m_i = \Delta x \times 0.1$, so the force required to lift it is

$$F_i = m_i g = \Delta x \times 0.1 \times g =$$

and the work required is

$$W_i = F_i x_i = \Delta x \times 0.1 \times g \times x_i$$
$$= 0.1\, g\, x_i\, \Delta x$$

we divide the rope into n such pieces, and sum up the work required:

$$W \approx \sum_{i=1}^{n} 100\, g\, x_i\, \Delta x \longrightarrow \int_0^{50} 0.1\, g\, x\, dx = W$$

$$= \left[0.05\, g\, x^2 \right]_0^{50}$$

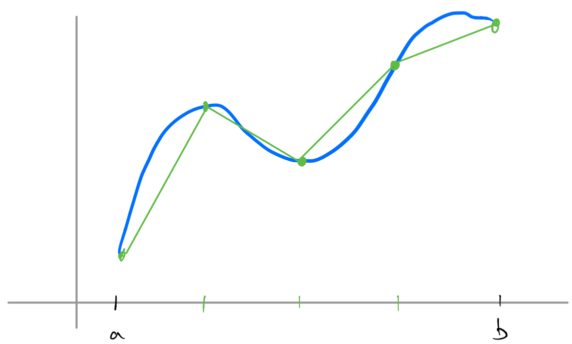
$$\approx 1225$$

Lengths of curves

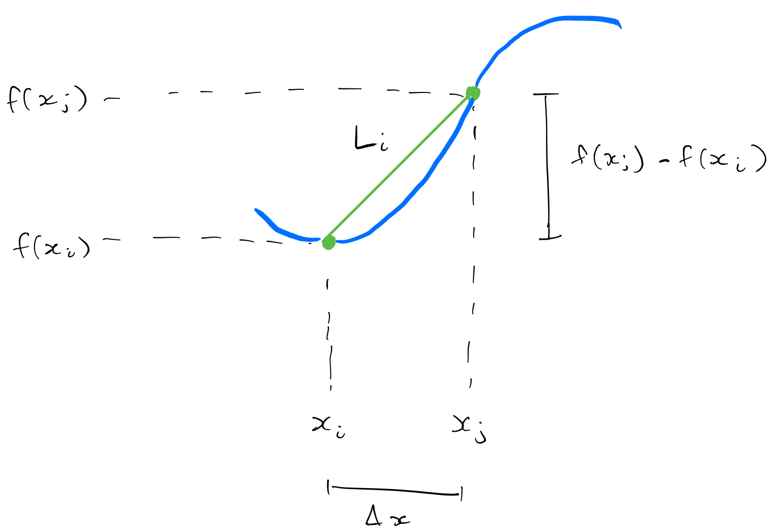
$f(x)$ continuously differentiable on $[a, b]$

We can find the length of the curve $\{(x, f(x)) : x \in [a, b]\}$ by setting up a Riemann integral.

Approximate with lengths of line segments



For a segment between x_i and x_j :



The length $L_i = \sqrt{\Delta x^2 + (f(x_j) - f(x_i))^2}$

Since $f(x)$ is continuously differentiable there is a theorem (called the Mean Value Theorem) which states that there exists c_i between x_i and x_j such that

$$f'(c_i) = \frac{f(x_j) - f(x_i)}{x_j - x_i}$$

i.e. $\quad f(x_j) - f(x_i) = f'(c_i)\,\Delta x$

substituting this into the expression for L_i:

$$L_i = \sqrt{\Delta x^2 + f'(c_i)^2 \Delta x^2} = \sqrt{1 + f'(c_i)^2}\ \Delta x$$

so the total length

$$\sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\ \Delta x$$

$$= \sum_{i=1}^{n} \sqrt{1 + f'(x_i)^2}\ \Delta x$$

(Recall that the Riemann integral actually doesn't depend on which value in the interval Δx is used in the Riemann sum - intuitively: as the width $\Delta x \to 0$, every point in the interval approaches x_i)

So now taking the limit $\begin{array}{c} n \to \infty \\ \Delta x \to 0 \end{array}$

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$