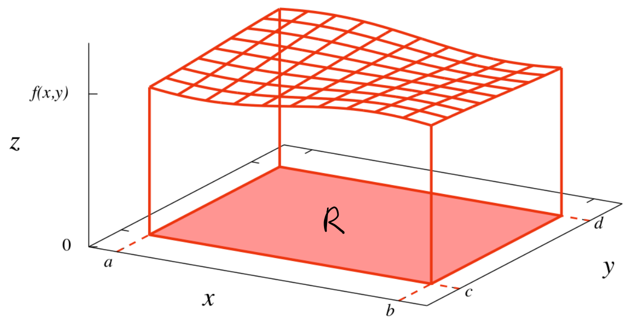
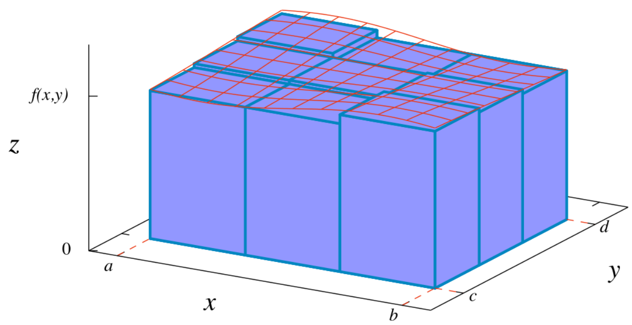
Double integrals

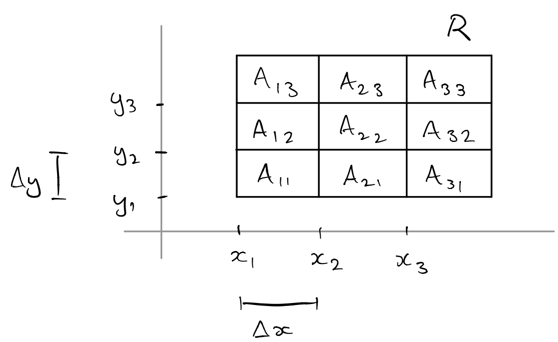
How can we find the volume below a surface $z = f(x,y)$ and above the rectangle $R = \{ (x,y) \in \mathbb{R}^2 : a \leq x \leq b \, , \, c \leq y \leq d \}$?



First approximate by summing up volumes of rectangular prisms



The plan is to take the limit as the number of rectangular prisms $\to \infty$ and their base area $\to 0$. Need to be able to write the sums.

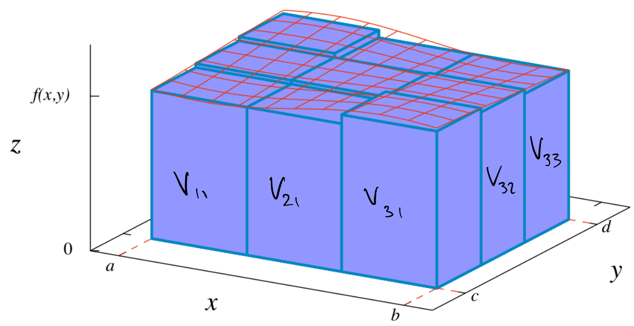


The area A_{ij} has bottom left corner at (x_i, y_j)

The total area can be written as a double sum

$$\sum_{i}^{3} \sum_{j}^{3} A_{ij} = \sum_{i}^{3} \left(A_{i1} + A_{i2} + A_{i3} \right) = \left(A_{11} + A_{12} + A_{13} \right) + \left(A_{21} + A_{22} + A_{23} \right) + \left(A_{31} + A_{32} + A_{33} \right)$$

write V_{11}, V_{12} ... for the volume above A_{11}, A_{12}, ...



Then $V_{11} = A_{11} \times f(x_1, y_1)$ because $f(x_1, y_1)$ is the height of V_{11}

$V_{21} = A_{21} \times f(x_2, y_1)$

$\vdots$

$V_{ij} = A_{ij} \times f(x_i, x_j)$

So the total volume is also a double sum

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{3} V_{ij} = \sum_{i}^{3} \sum_{j}^{3} A_{ij}\, f(x_i, x_j)$$

Since we have chosen equally spaced intervals along the x and y axes: $A_{11} = \Delta x \Delta y$ $\quad A_{21} = \Delta x \Delta y$ $\ldots$ $A_{ij} = \Delta x \Delta y$

and therefore

$$V \approx \sum_{i}^{3} \sum_{j}^{3} f(x_i, x_j)\, \Delta x \Delta y$$

Inspired by our previous success with single Riemann integrals we define the double integral over a rectangular region R

$$\iint_R f(x,y)\, dA = \lim_{\substack{m,n \to \infty \\ \Delta x, \Delta y \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta x \Delta y$$

if the limit exists. Here dA is called the area element.

Calculating double integrals

Theorem 7.2. *(Fubini's theorem for rectangular regions)*

Let $f(x,y)$ be a continuous function on the rectangular region R defined by $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$. Then the double integral $\iint\limits_R f(x,y)\,dA$ exists and

$$\iint\limits_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy \qquad \leftarrow \text{integrate wrt } x \text{ first, treating } y \text{ as constant}$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx. \qquad \leftarrow \text{integrate wrt } y \text{ first treating } x \text{ as const.}$$

EXAMPLE

$f(x,y) = x^2 + xy$, $R = \{ (x,y) : 1 \le x \le 2 , -1 \le y \le 1 \}$

$$\iint\limits_R f(x,y)\,dA = \int_{-1}^{1}\int_{1}^{2} x^2 + xy \, dx\, dy$$

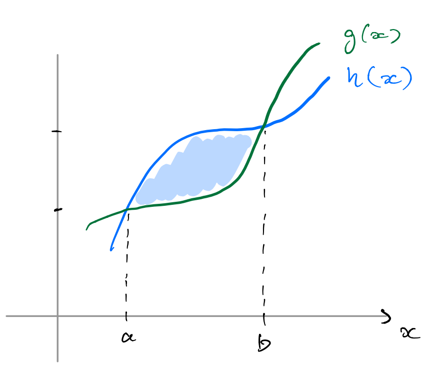
$$= \int_{-1}^{1} \left[\frac{x^3}{3} + \frac{x^2 y}{2} \right]_1^2 dy$$

$$= \int_{-1}^{1} \left[\frac{8}{3} + 2y - \frac{1}{3} - \frac{1}{2}y \right] dx$$

$$= \left[\frac{7}{3}y + \frac{3}{4}y^2 \right]_{-1}^{1}$$

$$= \frac{7}{3} + \frac{3}{4} + \frac{7}{3} - \frac{3}{4} = \frac{14}{3}$$

Double integrals over bounded regions

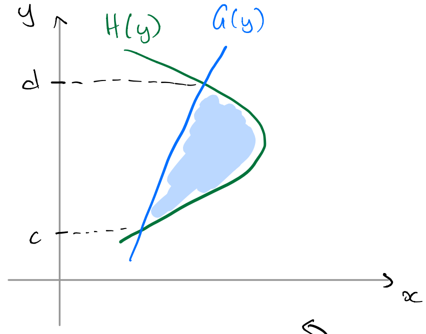


TYPE 1 : vertically simple region

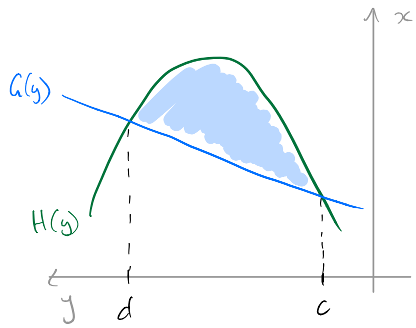
$R = \{ (x,y) : a \le x \le b , g(x) \le y \le h(x) \}$

$g(x) \le h(x)$ for all $a \le x \le b$

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$



Rotate



TYPE 2 : horizontally simple region

$$R = \{(x,y) : G(y) \le x \le H(y) ,\ c \le y \le d\}$$

where $G(y) \le H(y)$ for $c \le y \le d$

$$\iint_R f(x,y)\, dA = \int_c^d \int_{G(y)}^{H(y)} f(x,y)\, dx\, dy$$

Note: if $g(x)$ and $h(x)$ are invertible for $a \le x \le b$ then the TYPE 1 region

$$R = \{(x,y) : a \le x \le b ,\ g(x) \le y \le h(x)\}$$

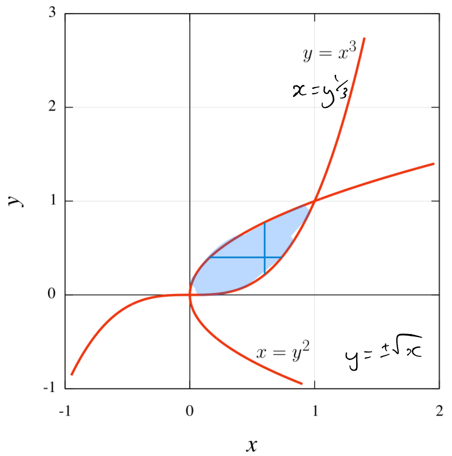
is also a TYPE 2 region

$$R = \left\{ (x,y) : c \le y \le d ,\ \begin{array}{ll} g^{-1}(y) \le x \le h^{-1}(y) & \text{if } g^{-1}(y) \le h^{-1}(y) \\ \text{OR} & \\ h^{-1}(y) \le x \le g^{-1}(y) & \text{if } h^{-1}(y) \le g^{-1}(y) \end{array} \right\}$$

sim. if $G(y)$ and $H(y)$ are invertible...

EXAMPLE Let $f(x,y) = 1$ and R the region bounded by $y = x^3$ and $x = y^2$. Evaluate $\iint_R f(x,y)\, dA$

$$R = \{(x,y) : 0 \le x \le 1 ,\ x^3 \le y \le \sqrt{x}\}$$
$$= \{(x,y) : 0 \le y \le 1 ,\ y^{1/3} \le x \le y^2\}$$



Two options

$$\iint_R f \, dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 1 \, dy \, dx$$

$$= \int_0^1 \int_{y^2}^{y^{1/3}} 1 \, dx \, dy$$

$$\int_0^1 \int_{x^3}^{\sqrt{x}} 1 \, dy \, dx = \int_0^1 \left[y \right]_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\sqrt{x} - x^3 \right) dx$$

$$= \left[\frac{2x^{3/2}}{3} - \frac{x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

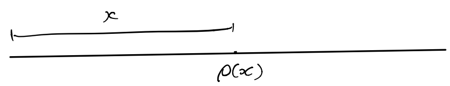
$$\int_0^1 \int_{y^2}^{y^{1/3}} 1 \, dx \, dy = \int_0^1 \left[x \right]_{y^2}^{y^{1/3}} dy$$

$$= \int_0^1 y^{\frac{1}{3}} - y^2 \, dy$$

$$= \left[\frac{3}{4} y^{4/3} - \frac{y^3}{3} \right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

The integral as a weighted sum

We have seen that integration in one variable is not just for finding areas. For example; consider a thin wire with varying density $\rho(x)$ kg/m



The mass of the wire is given by

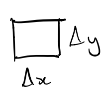
$$\sum_{i}^{n} \rho(x_i)\,\Delta x \longrightarrow \int_0^L \rho(x)\,dx = M$$

"weighted sum" - we are adding up small intervals weighted by their density.

The weighting need not be a density (of the kg/?) kind. It can be an area (eg: finding volumes of rotational solids), a probability density, electric charge density... depending on the application at hand.

Mental picture: a single integral is a (limit of a) weighted sum intervals. Similarly, a <u>double</u> integral is a (limit of a) weighted sum of areas

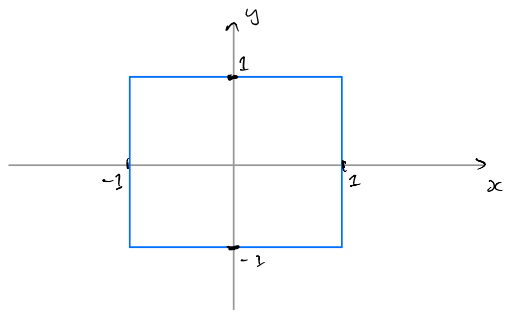
$$\sum_{i} \sum_{j} f(x_i, y_j)\,\Delta x\,\Delta y \longrightarrow \iint_R f(x,y)\,dA$$



and the weighting need not be a height (as in finding the volume under a surface) - it could be mass density, energy density, electric charge density, the speed of a fluid flowing through the region R...

In particular, if the weighting is $f(x,y) = 1$, we are just summing up areas, so $\iint_R 1\,dA = \text{Area}(R)$

Example consider a square metal plate



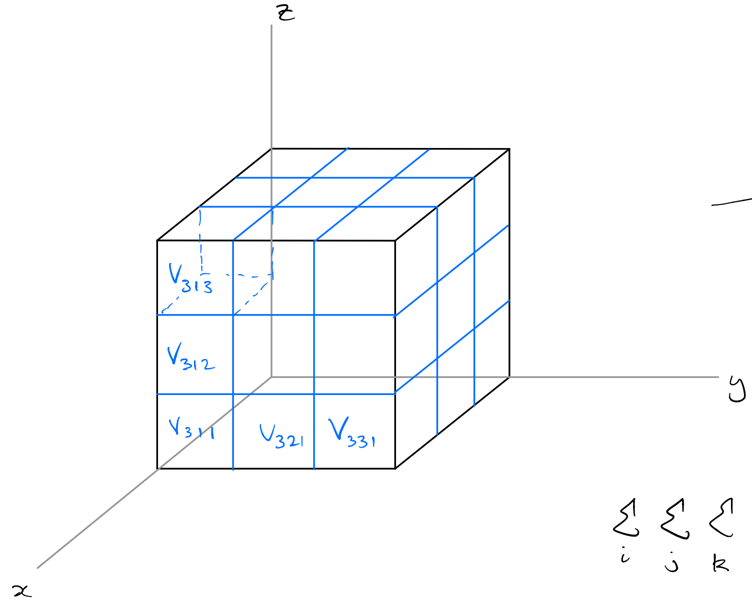
with density $\rho(x,y) = 1 + x^2 + y^2$ kg/m^2 i.e. density increasing with distance from $\underset{\sim}{0}$, maybe it gets thicker away from zero.

We get the mass of the plate by summing up small areas weighted by density:

$$\sum_i \sum_j \rho(x_i, x_j)\, \Delta x\, \Delta y \longrightarrow \int_{-1}^{1}\int_{-1}^{1} \rho(x,y)\, dA$$

Triple integrals.

	Riemann sums		Integrals
Single :	weighted sums of small lengths Δx	$\longrightarrow$	$\int f(x)\,dx$
double :	weighted sums of small areas $\Delta x \Delta y$		$\iint f(x,y)\,dx\,dy$
triple :	Weighted sums of volumes $\Delta x \Delta y \Delta z$		$\iiint f(x,y,z)\,dx\,dy\,dz$



$$\xrightarrow{\;f\;} \mathbb{R}$$

$$f(x,y,z)$$

$$\sum_i \sum_j \sum_k f(x_i, y_j, z_k)\,\Delta x \Delta y \Delta z$$

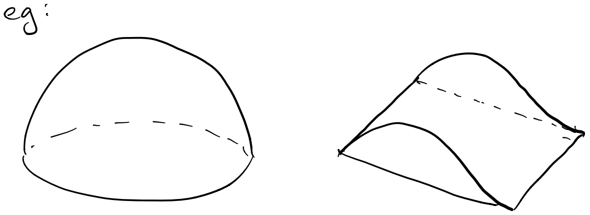
Let $f(x,y,z)$ be a scalar function which is bounded on a solid $R \subset \mathbb{R}^3$

The triple integral of f over R is

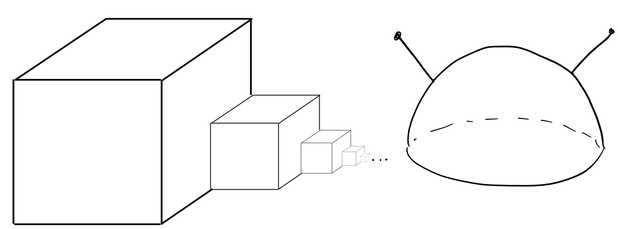
$$\iiint_R f(x,y,z)\,dV = \lim_{\substack{l,m,n\to\infty \\ \Delta x, \Delta y, \Delta z \to 0}} \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j, z_k)\,\Delta x_i \Delta y_j \Delta z_k$$

Note: by solid we mean a bounded subset $R \subset \mathbb{R}^3$ whose boundary ∂R is a finite union of continuously differentiable surfaces.

eg:



solids



not solids

If $f(x,y,z)=1$ we get the volume of the region of integration

$$\iiint_R 1 \, dV = \text{Volume}(R)$$

Fubini's theorem for boxes

$f(x,y,z)$ bounded function on $B = \{(x,y,z) \in \mathbb{R}^3 : a \le x \le b, \ c \le y \le d, \ p \le z \le q\}$

then
$$\iiint_B f \, dV = \int_a^b \int_c^d \int_p^q f(x,y,z) \, dz \, dy \, dx$$

$$= \int_a^b \int_p^q \int_c^d f(x,y,z) \, dy \, dz \, dx$$

$$= \dots$$ i.e. all permutations of $dx\,dy\,dz$ give the same result.

For more general regions, eg:

$$R_1 = \{(x,y,z) : a \le x \le b, \quad c \le y \le d, \quad g(x,y) \le z \le h(x,y)\}$$

$$R_2 = \{(x,y,z) : A(y) \le x \le B(y), \quad G(x,z) \le y \le H(x,z), \quad p \le z \le q\}$$

as with double integrals, the order of integration must be chosen carefully so that the result is a number not a function, and all variables are integrated!

i.e.
$$\iiint_{R_1} f \, dV = \int_a^b \int_c^d \int_{g(x,y)}^{h(x,y)} f \quad dz \, dy \, dx$$

$$= \int_c^d \int_a^b \int_{g(x,y)}^{h(x,y)} f \quad dz \, dx \, dy$$

$$\iiint_{R_2} f \, dV = \int_p^q \int_{A(y)}^{B(y)} \int_{G(x,z)}^{H(x,z)} f \quad dy \, dx \, dz$$

$$\neq \int_{p}^{q} \int_{G(x,z)}^{H(x,z)} \int_{A(y)}^{B(y)} f \, dx \, dy \, dz$$

Triple integrals - examples

Integrate $f(x,y,z) = x+y+z$ over the region

$$R = \{ (x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 2 \}$$

$$\iiint_R f(x,y,z)\, dV = \int_0^1 \int_0^1 \int_0^2 x+y+z \; dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 \left[xz + yz + \frac{z^2}{2} \right]_{z=0}^{z=2} dy\, dx$$

$$= \int_0^1 \int_0^1 2x + 2y + 2 \; dy\, dx$$

$$= \int_0^1 \left[2xy + y^2 + 2y \right]_{y=0}^{y=1} dx$$

$$= \int_0^1 2x + 3 \; dx$$

$$= \left[x^2 + 3x \right]_0^1$$

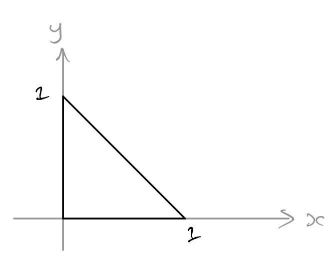
$$= 4$$

Evaluate $\iiint_T z\, dV$ where T is the solid bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

SKETCH!

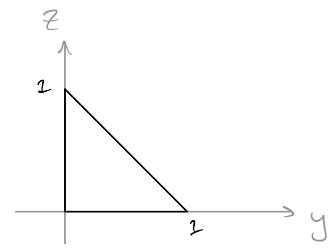
in the x,y plane $(z=0)$ so

$$x+y=1$$
$$y = 1-x$$



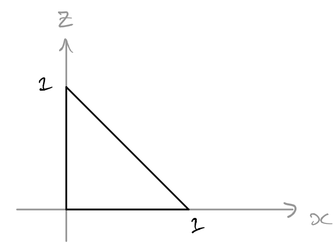
in the y,z plane ($x=0$):

$$y+z=1$$
$$z=1-y$$

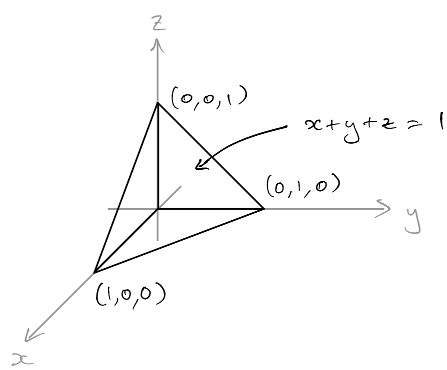


In the x,z plane ($y=0$):

$$x+z=1$$
$$z=1-x$$



putting it all together:



limits $\quad 0 \le x \le 1, \quad 0 \le y \le 1-x, \quad 0 \le z \le 1-x-y$

(not the only way to do it)

integral

$$\iiint_T z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_{z=0}^{z=1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)^2\,dy\,dx$$

$$= \int_0^1 \left[\frac{-1}{6}(1-x-y)^3 \right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 -\frac{1}{6}\left(1 - x - (1-x)\right)^3 + \frac{1}{6}(1-x)^3 \, dx$$

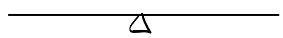
$$= \int_0^1 \frac{1}{6}(1-x)^3 \, dx$$

$$= \left[\frac{(1-x)^4}{24} \right]_0^1 = \frac{1}{24}$$

Centre of mass

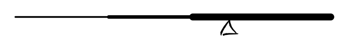
centre of mass of a piece of wire is the point where an applied force produces no rotation (balancing point)
(torque)

uniform density



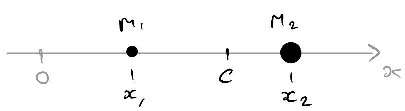
centre of mass = geometric centre

density increasing $\longrightarrow$



centre of mass moves away from the middle.

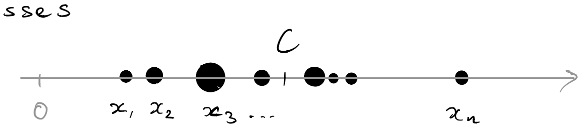
for a pair of point masses



the centre of mass C satisfies $\quad m_1(C - x_1) = m_2(x_2 - C)$

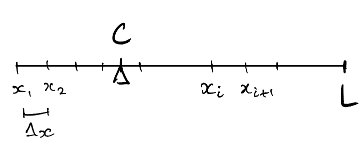
or
$$m_1(C - x_1) + m_2(C - x_2) = 0$$
$$(m_1 + m_2)C = m_1 x_1 + m_2 x_2$$
$$C = \frac{m_1 x_1 + m_2 x_2}{M}, \quad M = m_1 + m_2$$

For n point masses



$$\sum_i m_i(C - x_i) = 0$$
$$\sum_i m_i C = \sum_i m_i x_i \qquad \text{average position} = \frac{1}{n}\sum_i x_i$$
$$C = \frac{1}{M}\sum_i m_i x_i \quad \leftarrow \text{mass-weighted average of position}$$

we can approximate a wire with non-uniform density $\rho(x)$ by dividing it into intervals



and treating the section of wire $[x_i, x_{i+1}]$ as a point mass at x_i with mass $\rho(x_i)\Delta x$. Then the centre of mass is

$$C \simeq \frac{1}{M}\sum_i x_i\,\rho(x_i)\,\Delta x$$

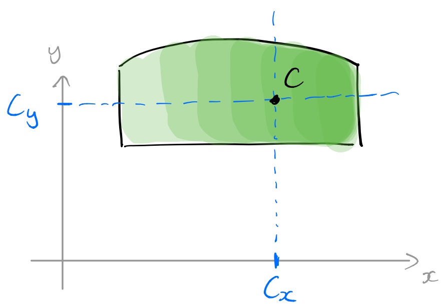
where the total mass is

$$M = \sum_i \rho(x_i)\Delta x$$

Taking the limit of these Riemann sums:

$$C = \frac{1}{M}\int_0^L x\,\rho(x)\,dx \qquad \text{where} \quad M = \int_0^L \rho(x)\,dx$$

Centre of mass of a 2D object R with density $\rho(x,y)$



balancing lines in x and y directions
$C = (C_x, C_y)$ is their intersection.

to find C_x, we need the mass-weighted average of x position over the whole object:

$$C_x = \frac{1}{M}\iint_R x\,\rho(x,y)\,dA \quad , \quad \text{where} \quad M = \iint_R \rho(x,y)\,dA$$

C_y is the weighted average of y-position

$$C_y = \frac{1}{M}\iint_R y\,\rho(x,y)\,dA$$

And the centre of mass is the point $C = (C_x, C_y)$

For a 3D object $C = (C_x, C_y, C_z)$

$$C_x = \frac{1}{M} \iiint_R x \rho(x,y,z)\, dV$$

$$C_y = \frac{1}{M} \iiint_R y \rho(x,y,z)\, dV$$

$$C_z = \frac{1}{M} \iiint_R z \rho(x,y,z)\, dV$$

$$M = \iiint_R \rho(x,y,z)\, dV$$