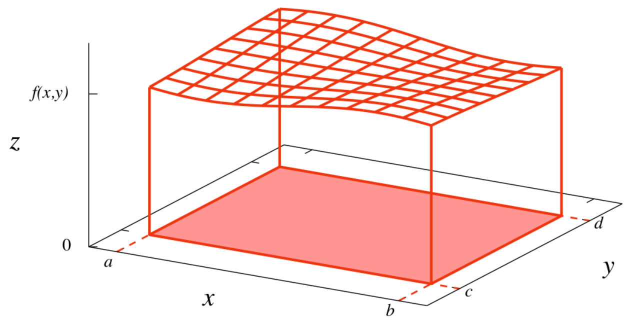
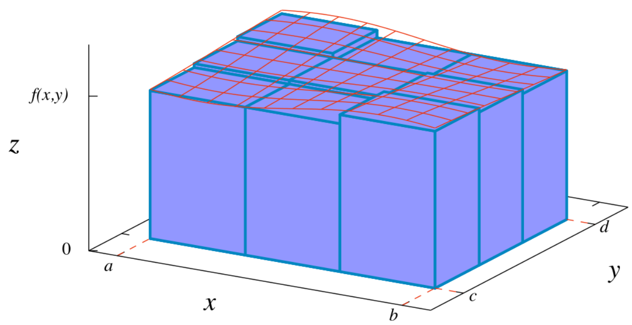
Double integrals

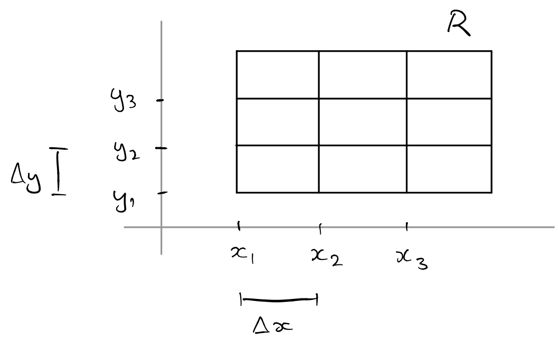
How can we find the volume below a surface $z = f(x,y)$ and above the rectangle $R = \{ (x,y) \in \mathbb{R}^2 : a \le x \le b, c \le y \le d \}$?



First approximate by summing up volumes of rectangular prisms



The plan is to take the limit as the number of rectangular prisms $\to \infty$ and their base area $\to 0$. Need to be able to write the sums.

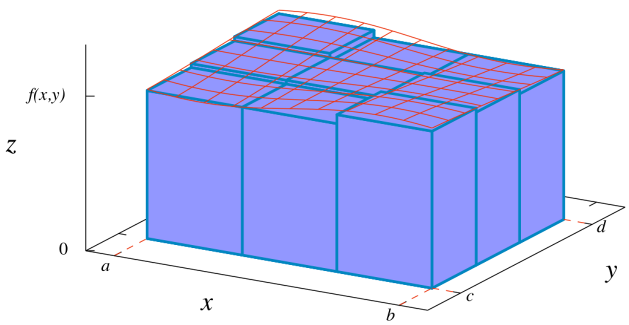


The area has bottom left corner at

The total area can be written as a <u>double sum</u>

$$\sum_{i}^{3} \sum_{j}^{3} A_{ij} = \qquad =$$

Write V_{11}, V_{12} ... for the volume above $A_{11}, A_{12}, \ldots$



Then $V_{11} =$ because $f(x_1, y_1)$ is the height of V_{11}

$V_{21} =$

$\vdots$

$V_{ij} =$

So the total volume is also a double sum

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{3} \qquad = \sum_{i}^{3} \sum_{j}^{3}$$

Since we have chosen equally spaced intervals along the x and y axes: $A_{11} = \Delta x \Delta y \quad A_{21} = \Delta x \Delta y \quad \ldots \quad A_{ij} =$

and therefore

$$V \approx \sum_{i}^{3} \sum_{j}^{3}$$

Inspired by our previous success with single Riemann integrals we define the double integral over a rectangular region R

$$\iint_R f(x,y)\, dA = \lim_{\substack{m,n \to \infty \\ \Delta x, \Delta y \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta x \Delta y$$

if the limit exists. Here dA is called the area element.

Calculating double integrals

Theorem 7.2. *(Fubini's theorem for rectangular regions)*

Let $f(x,y)$ be a continuous function on the rectangular region R defined by $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$. Then the double integral $\iint\limits_R f(x,y)\,dA$ exists and

$$\iint\limits_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

← integrate wrt x first, treating y as constant

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

← integrate wrt y first treating x as const.

EXAMPLE

$f(x,y) = x^2 + xy$, $R = \{(x,y) : 1 \le x \le 2, -1 \le y \le 1\}$

$\iint\limits_R f(x,y)\,dA =$

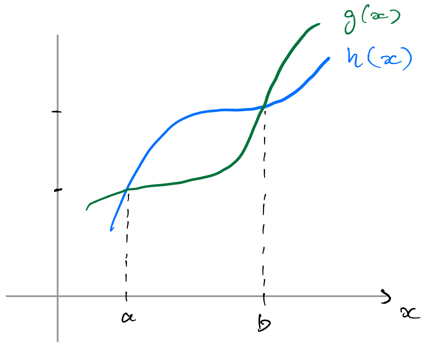
$=$

$=$

$=$

$=$ $\quad = \frac{14}{3}$

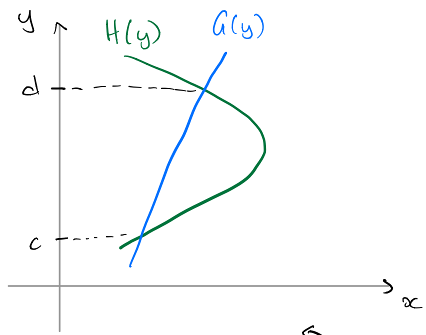
Double integrals over bounded regions



TYPE 1 : vertically simple region

$R = \{(x,y) : a \le x \le b, g(x) \le y \le h(x)\}$

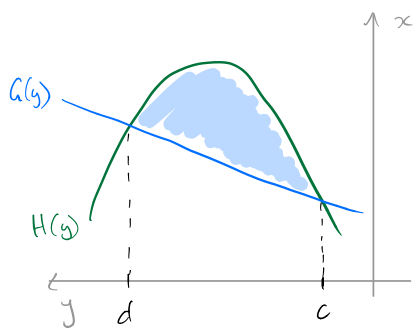
$\iint\limits_R f(x,y)\,dA =$



TYPE 2 : horizontally simple region

$$R = \{(x,y) : G(y) \le x \le H(y),\ c \le y \le d\}$$

Rotate ↻



$$\iint_R f(x,y)\,dA =$$

Note: if $g(x)$ and $h(x)$ are invertible for $a \le x \le b$ then the TYPE 1 region

$$R = \{(x,y) : a \le x \le b,\ g(x) \le y \le h(x)\}$$

is also a TYPE 2 region

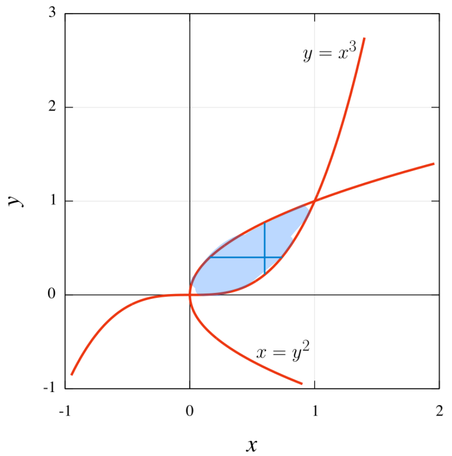
$$R = \left\{(x,y) : c \le y \le d,\ \begin{array}{ll} g^{-1}(y) \le x \le h^{-1}(y) & \text{if } g^{-1}(y) \le h^{-1}(y) \\ \text{OR} & \\ h^{-1}(y) \le x \le g^{-1}(y) & \text{if } h^{-1}(y) \le g^{-1}(y) \end{array}\right\}$$

sim. if $G(y)$ and $H(y)$ are invertible...

EXAMPLE Let $f(x,y) = 1$ and R the region bounded by $y = x^3$ and $x = y^2$. Evaluate $\iint_R f(x,y)\,dA$

$$R = \{(x,y) : \qquad\qquad\qquad \}$$

$$= \{(x,y) : \qquad\qquad\qquad \}$$



Two options

$$\iint_R f\, dA =$$

$$=$$

$$\int_0^1 \int_{x^3}^{\sqrt{x}} 1 \, dy\, dx = \int_0^1$$

$$= \int_0^1$$

$$= \Big[\qquad \Big]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

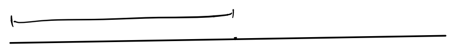
$$\int_0^1 \int_{y^2}^{y^{1/3}} 1 \, dx\, dy = \int_0^1 \big[x \big]_{y^2}^{y^{1/3}} dy$$

$$= \int_0^1 y^{\frac{1}{3}} - y^2 \; dy$$

$$= \Big[\frac{3}{4} y^{4/3} - \frac{y^3}{3} \Big]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

The integral as a weighted sum

We have seen that integration in one variable is not just for finding areas. For example, consider a thin wire with varying density $\rho(x)$ kg/m



The mass of the wire is given by

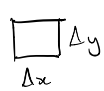
$$\sum_{i}^{n} \qquad \longrightarrow \qquad = M$$

"weighted sum" - we are adding up small intervals weighted by their density.

The weighting need not be a density of the kg/? kind. It can be an area (eg: finding volumes of rotational solids), a probability density, electric charge density... depending on the application at hand.

Mental picture: a single integral is a (limit of a) weighted sum intervals. Similarly, a <u>double</u> integral is a (limit of a) weighted sum of areas

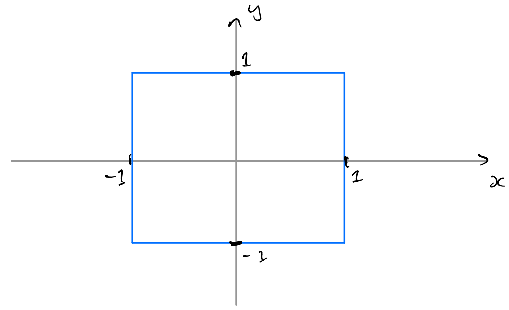
$$\sum_i \sum_j f(x_i, y_j)\, \Delta x\, \Delta y \longrightarrow \iint_R f(x,y)\, dA$$



and the weighting need not be a height (as in finding the volume under a surface) - it could be mass density, energy density, electric charge density, the speed of a fluid flowing through the region R...

In particular, if the weighting is $f(x,y) = 1$, we are just summing up areas, so $\iint_R 1\, dA = \text{Area}(R)$

Example consider a square metal plate



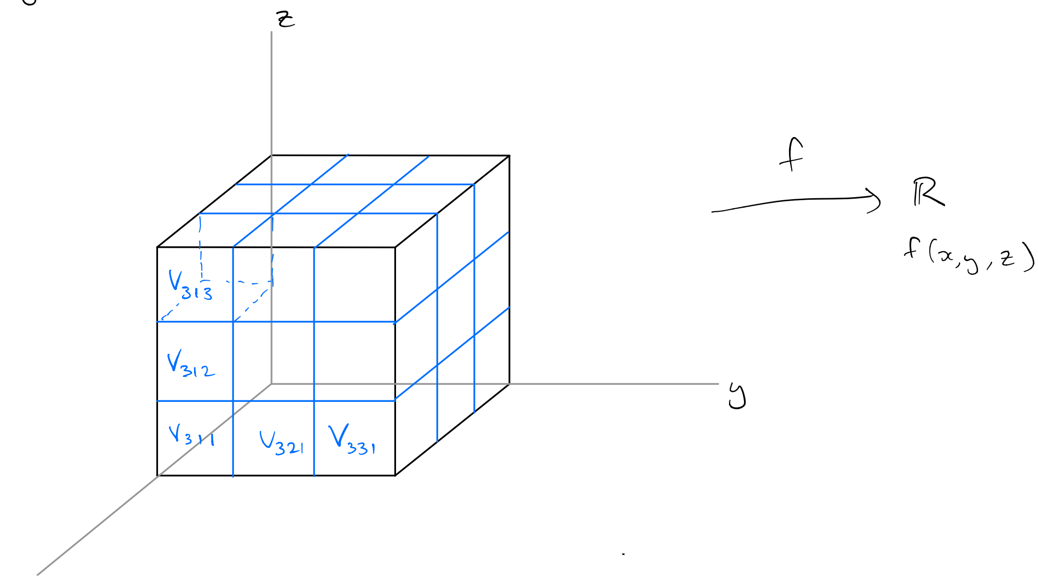
with density $\rho(x,y) = 1 + x^2 + y^2$ kg/m^2 i.e. density increasing with distance from $\underset{\sim}{0}$, maybe it gets thicker away from zero.

We get the mass of the plate by summing up small areas weighted by density:

$$\sum_i \sum_j \rho(x_i, x_j)\, \Delta x \Delta y \longrightarrow \int_{-1}^{1}\int_{-1}^{1} \rho(x,y)\, dA$$

Triple integrals.

	Riemann sums		Integrals
Single :	weighted sums of small lengths	→	$\int f(x)\,dx$
double :	weighted sums of small areas		$\iint f(x,y)\,dx\,dy$
triple :	Weighted sums of volumes		



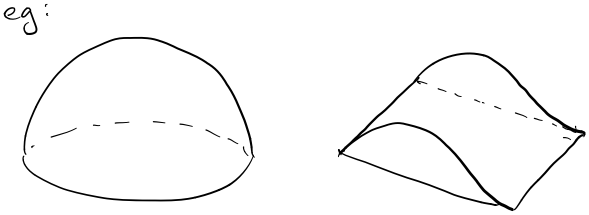
Let $f(x,y,z)$ be a scalar function which is bounded on a solid $R \subset \mathbb{R}^3$

The triple integral of f over R is

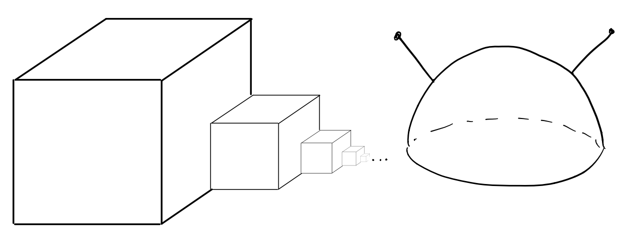
$$\iiint_R f(x,y,z)\,dV = \lim_{\substack{l,m,n\to\infty \\ \Delta x,\Delta y,\Delta z \to 0}} \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i, y_j, z_k)\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

Note: by solid we mean a bounded subset $R \subset \mathbb{R}^3$ whose boundary ∂R is a finite union of continuously differentiable surfaces.

eg:



solids



not solids

If $f(x,y,z) = 1$ we get the volume of the region of integration

$$\iiint_R 1 \, dV = \text{Volume}(R)$$

Fubini's theorem for boxes

$f(x,y,z)$ bounded function on $B = \{ (x,y,z) \in \mathbb{R}^3 : a \le x \le b, \ c \le y \le d, \ p \le z \le q \}$

then

$$\iiint_B f \, dV = \int_a^b \int_c^d \int_p^q f(x,y,z) \, dz \, dy \, dx$$

$$= \int_a^b \int_p^q \int_c^d f(x,y,z) \, dy \, dz \, dx$$

$$= \dots$$

i.e. all permutations of $dx\,dy\,dz$ give the same result.

For more general regions, eg:

$$R_1 = \{ (x,y,z) : \ a \le x \le b, \ c \le y \le d, \ g(x,y) \le z \le h(x,y) \}$$

$$R_2 = \{ (x,y,z) : \ A(y) \le x \le B(y), \ G(x,z) \le y \le H(x,z), \ p \le z \le q \}$$

as with double integrals, the order of integration must be chosen carefully so that the result is a number not a function and all variables are integrated!

i.e. $$\iiint_{R_1} f \, dV =$$

$$=$$

$$\iiint_{R_2} f \, dV =$$

≠

Triple integrals – examples

Integrate $f(x,y,z) = x+y+z$ over the region

$$R = \{ (x,y,z) : 0 \le x \le 1 , \ 0 \le y \le 1 , \ 0 \le z \le 2 \}$$

$$\iiint_R f(x,y,z)\, dV =$$

$=$

$=$

$=$

$=$

$=$

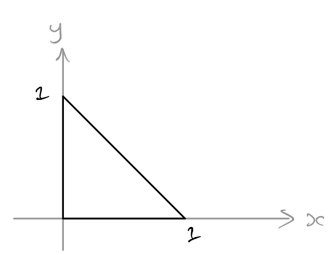
$=$

Evaluate $\iiint_T z \, dV$ where T is the solid bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

SKETCH!

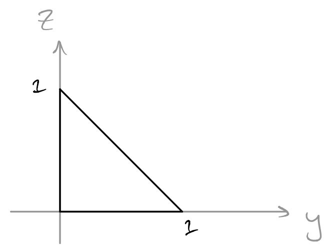
in the x,y plane ($z=0$) so

$$x+y=1$$
$$y = 1-x$$



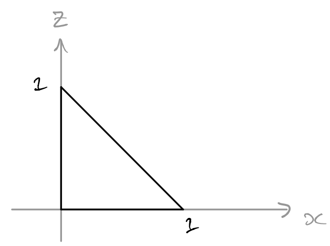
in the y, z plane ($x=0$):

$$y + z = 1$$
$$z = 1 - y$$

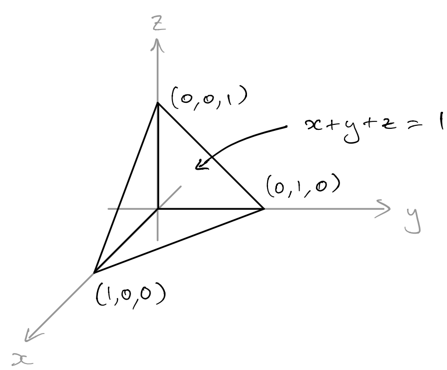


In the x, z plane ($y=0$):

$$x + z = 1$$
$$z = 1 - x$$



putting it all together:



limits

(not the only way to do it)

integral

$$\iiint_T z \, dV =$$

$$=$$

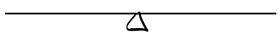
$$=$$

$$=$$

Centre of mass

centre of mass of a piece of wire is the point where an applied force produces no rotation (balancing point)
(torque)

uniform density

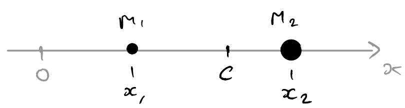


centre of mass = geometric centre

density increasing →

centre of mass moves away from the middle.

for a pair of point masses



the centre of mass C satisfies $m_1(C - x_1) = m_2(x_2 - C)$

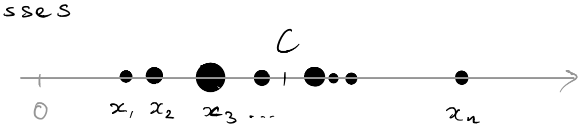
or

$= 0$

$=$

$C =$, $M = m_1 + m_2$

For n point masses

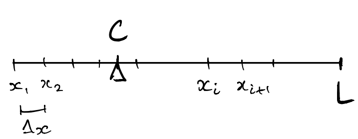


$=$

$=$ average position =

$c =$ mass-weighted average of position

we can approximate a wire with non-uniform density $\rho(x)$ by dividing it into intervals



and treating the section of wire $[x_i, x_{i+1}]$ as a point mass at x_i with mass $\rho(x_i)\Delta x$. Then the centre of mass is

$$C \simeq \frac{1}{M} \sum_i x_i \, \rho(x_i) \Delta x$$

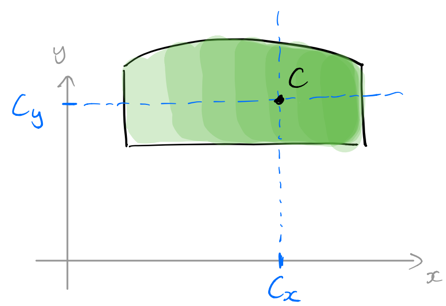
where the total mass is

$$M = \sum_i \rho(x_i) \Delta x$$

Taking the limit of these Riemann sums:

$$C = \frac{1}{M} \int_0^L x \rho(x) \, dx \qquad \text{where} \quad M = \int_0^L \rho(x) \, dx$$

Centre of mass of a 2D object R with density $\rho(x,y)$



balancing lines in x and y directions

$C = (C_x, C_y)$ is their intersection.

to find C_x, we take the mass-weighted average of x position over the whole object:

$$C_x = \qquad\qquad\qquad , \quad \text{where} \quad M = \iint_R \rho(x,y) \, dA$$

C_y is the weighted average of y-position

$$C_y = \frac{1}{M} \iint_R y \, \rho(x,y) \, dA$$

the centre of mass is the point $C = (C_x, C_y)$

For a 3D object $C = (C_x, C_y, C_z)$

$$C_x = \frac{1}{M} \iiint_R x \rho(x,y,z)\, dV$$

$$C_y = \frac{1}{M} \iiint_R y \rho(x,y,z)\, dV$$

$$C_z = \frac{1}{M} \iiint_R z \rho(x,y,z)\, dV$$

$$M = \iiint_R \rho(x,y,z)\, dV$$