Change of coordinates in double integrals

Recall that the idea behind an integration by substitution in single-variable calculus is to replace a complicated integral involving x by a simpler integral involving a new variable u which is a function of x, so that $x = f(u)$.

In particular we have

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\left(\frac{dg}{du}\right)\,du,$$

with the substitution:

$$x = g(u),$$

$$dx = g'(u)\,du.$$

In many cases the evaluation of a double integral

$$\iint_R f(x,y)\,dx\,dy,$$

over a certain region R in $\mathbb{R}^2$ can be significantly simplified by a change of coordinates. This is done by using some new coordinates e.g. u, v and a certain transformation $\mathbf{g}(u,v) = (x,y)$ which tells us how the new coordinates u, v are related to the old ones x, y. We write such a change of coordinates as

$$\begin{aligned} (x,y) &= \mathbf{g}(u,v) = (\phi(u,v), \psi(u,v)), \\ dx\,dy &= \left|\det\left(\frac{\partial \mathbf{g}(u,v)}{\partial(u,v)}\right)\right| du\,dv, \\ S &= \mathbf{g}^{-1}(R), \end{aligned}$$

where S is the new region of integration, i.e. the domain of the new coordinates u, v.

The matrix $\left(\dfrac{\partial \mathbf{g}(u,v)}{\partial(u,v)}\right)$ is the *Jacobian matrix* of the transformation $\mathbf{g}(u,v)$ at (u,v), that is

$$\left(\frac{\partial \mathbf{g}(u,v)}{\partial(u,v)}\right) = \begin{bmatrix} \dfrac{\partial \phi}{\partial u}(u,v) & \dfrac{\partial \phi}{\partial v}\partial(u,v) \\ \dfrac{\partial \psi}{\partial u}(u,v) & \dfrac{\partial \psi}{\partial v}(u,v) \end{bmatrix},$$

and its determinant $J(u,v)$ is called the *Jacobian* of $\mathbf{g}(u,v)$ at (u,v).

Then

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_S f(\mathbf{g}(u,v)) \left|\det\left(\frac{\partial \mathbf{g}(u,v)}{\partial(u,v)}\right)\right| du\,dv,$$

where S is just a description of the domain of integration in terms of u,v instead of x,y.

Note that the $|\cdot|$ in the change of variable formula means absolute value, and not the determinant – the Jacobian is already defined as a determinant!

See the Unit Reader for an explanation as to why the Jacobian $J(u, v)$ appears in the formula for change of variables, but observe the similarities between this and the single-variable case:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\left(\frac{dg}{du}\right)\,du.$$

Example

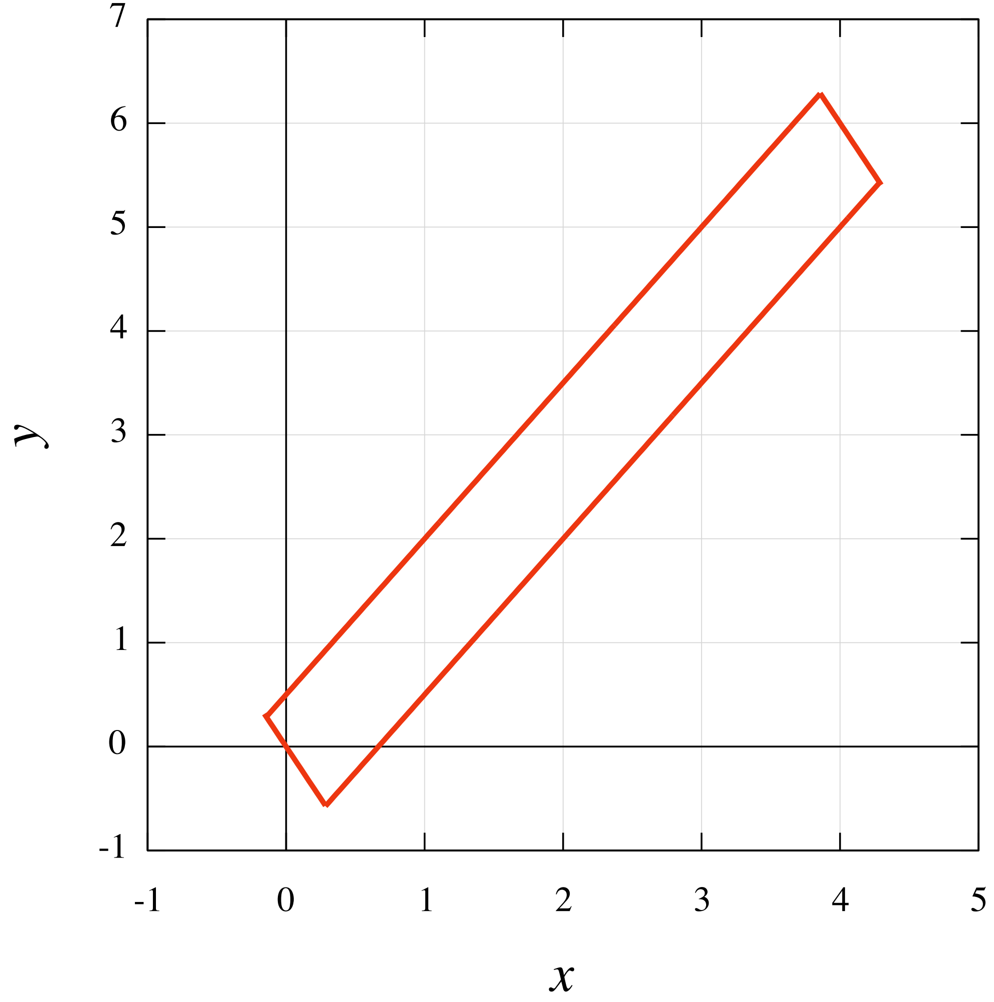
To evaluate the double integral

$$\iint_D 3x - 2y\,dA,$$

over the parallelogram D bounded by the lines

$$3x - 2y = 2 \quad , \quad 3x - 2y = -1 \quad , \quad 2x + y = 14 \quad , \quad 2x + y = 0,$$

we see that the difficulty with this problem is that the boundaries of the region do not align conveniently with the coordinate axes:



Guided by the fact that a linear change of variables maps parallelograms in one coordinate system to parallelograms in the other, it would seem that putting $u = 3x - 2y$ and $v = 2x + y$ would be a good choice.

Then the domain in (u, v) space is defined by $-1 \le u \le 2$ and $0 \le v \le 14$.

To calculate the Jacobian we first need to find x and y in terms of u and v; it is easy to show that with u and v defined as above then

$$x = \frac{1}{7}(u + 2v) \quad \text{and} \quad y = \frac{1}{7}(3v - 2u).$$

Then

$$\det \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \dfrac{1}{7} & \dfrac{2}{7} \\[2ex] -\dfrac{2}{7} & \dfrac{3}{7} \end{bmatrix} = \frac{1}{7} > 0.$$

Then it follows that

$$\iint_D 3x - 2y\, dA = \int_0^{14}\int_{-1}^{2} (u) \cdot \frac{1}{7}\, du\, dv = \frac{1}{7}\int_0^{14} \left[\frac{1}{2}u^2\right]_{-1}^{2} dv = \frac{3}{14}\int_0^{14} dv = 3.$$

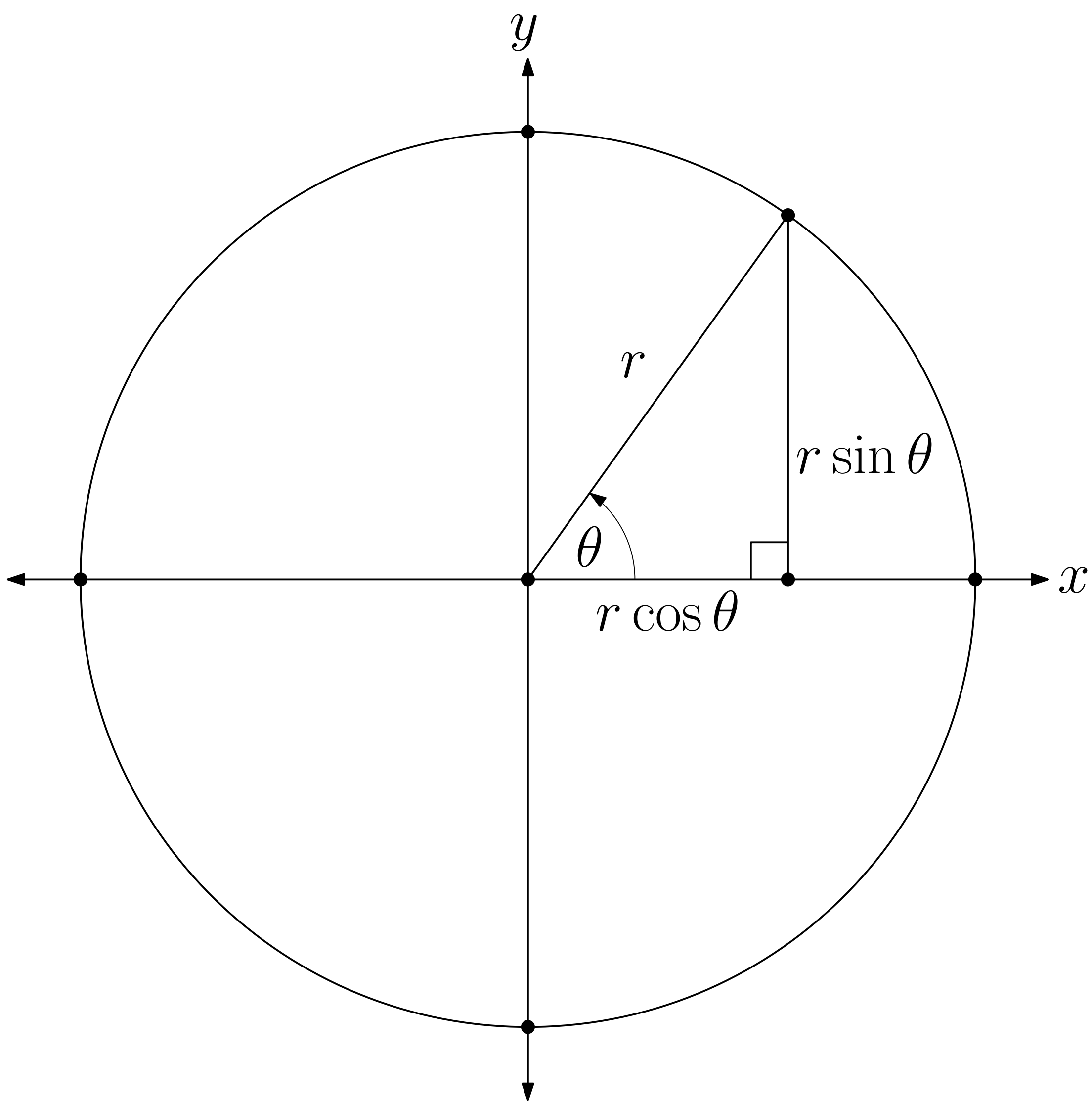
Example

Recall the definition of *polar coordinates*:

Let $x = r\cos\theta$ and $y = r\sin\theta$ for $r \geq 0$ and $\theta \in [0, 2\pi)$.

Then r, θ are called the polar coordinates of the point (x, y).

Notice that $r^2 = x^2 + y^2$, so $r = \sqrt{x^2 + y^2}$ and is the distance of the point (x, y) from the origin.



y
x
r
θ
r sin θ
r cos θ

Using the proposed transformation:

$$\mathbf{g}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

we see that:

$$\left|\det\left(\frac{\partial \mathbf{g}(r,\theta)}{\partial(r,\theta)}\right)\right| = \left|\det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right| = \left|r(\cos^2\theta + \sin^2\theta)\right| = r.$$

Then we have

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

We will use this change of coordinates every time the region R is 2-dimensional and has some sort of rotational symmetry.

Example set 3 – week 9

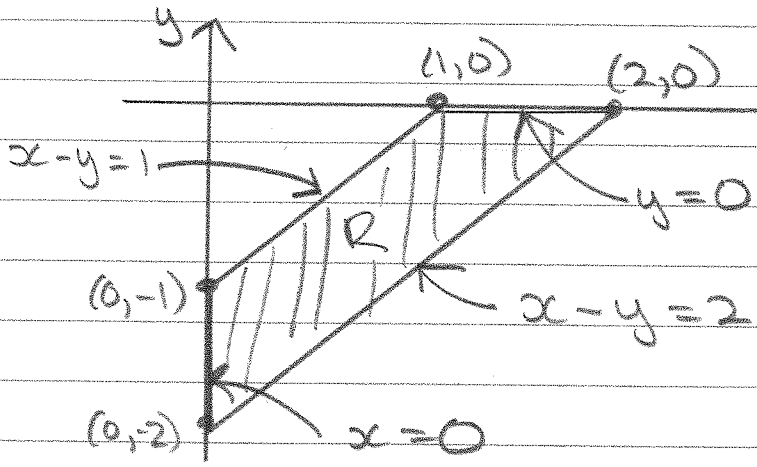
Example set 3

① Evaluate

$$\iint_R \exp\left(\frac{x+y}{x-y}\right) dA$$

where R is the trapezoidal region with vertices (1,0), (2,0), (0,−2) and (0,−1).

Note that $\exp(x) \equiv e^x$.



Let $u = x+y$, $v = x-y$

Then $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$

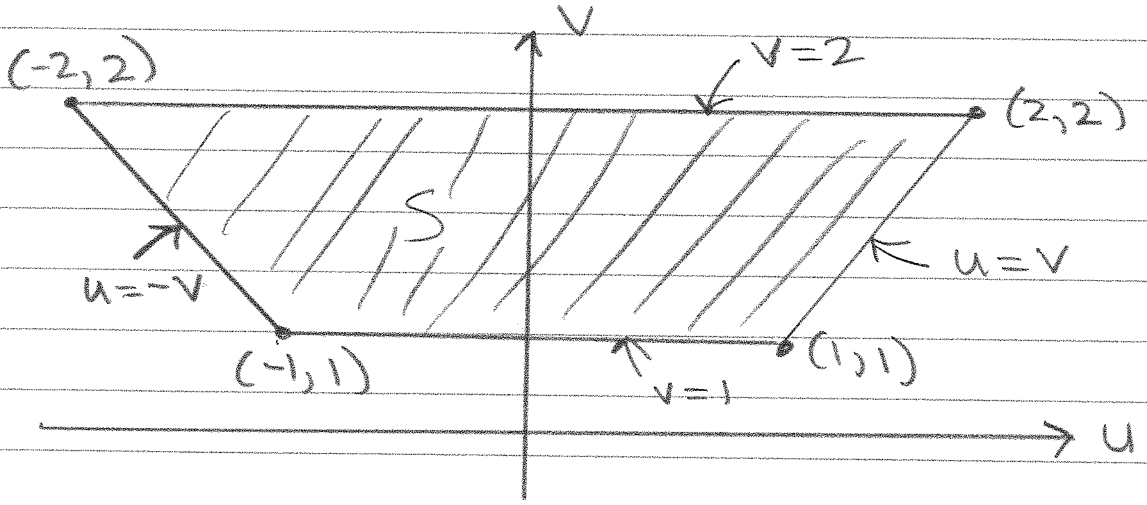
Now, when $y=0$ we have $1 \leq x \leq 2$
and $u=x$, $v=y \Rightarrow u=v$, $1 \leq u \leq 2$, $1 \leq v \leq 2$.

When $x-y=2$, $0 \leq x \leq 2$, $-2 \leq y \leq 0$
and $u = 2x-2$, $v=2 \Rightarrow -2 \leq u \leq 2$, $v=2$

When $x = 0$, $-2 \leq y \leq -1 \Rightarrow u = y$, $v = -y$
and $u = -v$, $-2 \leq u \leq -1$, $1 \leq v \leq 2$.

When $x - y = 1$, $0 \leq x \leq 1$, $-1 \leq y \leq 0$
and $u = 2x + 1$, $v = 1 \Rightarrow -1 \leq u \leq 1$, $v = 1$.

So the trapezoidal region in the xy-plane is transformed into the region shown below in the uv-plane:



Now, $\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = -\frac{1}{2}$

and $\left| \det J \right| = +\frac{1}{2}$

So $\iint_R \exp\left(\frac{x+y}{x-y}\right) dA = \int_1^2 \int_{-v}^{v} e^{u/v} \cdot \frac{1}{2} \cdot du\,dv$

$$= \int_1^2 \left[\frac{1}{2} v e^{\frac{u}{v}} \right]_{-v}^{v} dv$$

$$= \int_1^2 \left(\tfrac{1}{2} v e^{+1}\right) - \left(\tfrac{1}{2} v e^{-1}\right) dv$$

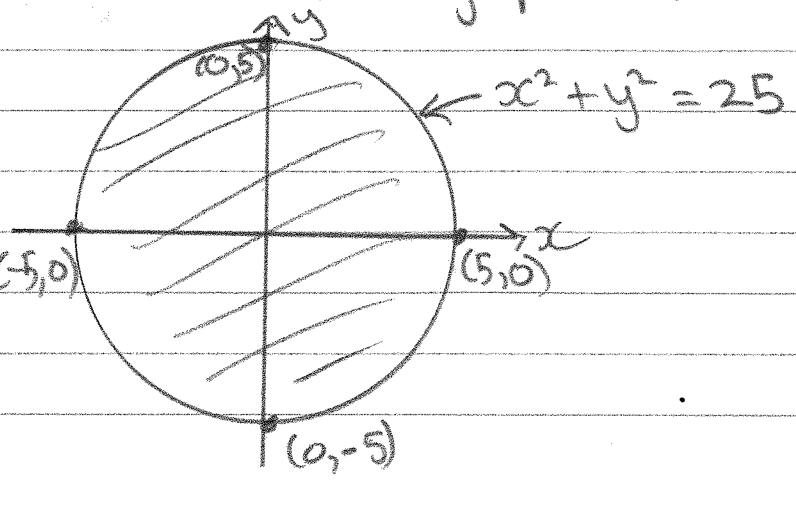
$$= \int_1^2 \left(\frac{e+e^{-1}}{2}\right) v \, dv$$

$$= \left[\left(\frac{e+e^{-1}}{2}\right) \cdot \frac{1}{2} v^2 \right]_1^2$$

$$= \frac{3}{4}\left(e-e^{-1}\right).$$

(2) Find the volume of a solid bounded below by the xy-plane and above by the paraboloid $z = 25 - x^2 - y^2$.

We could think of this as a double integral of $25 - x^2 - y^2$ over the circle $x^2 + y^2 = 25$ in the xy-plane:



So we have

$$\iint_R 25 - x^2 - y^2 \, dA$$

where $R = \{(x, y) : -5 \le x \le 5, \; -\sqrt{25-x^2} \le y \le \sqrt{25-y^2}\}$

So $V = \displaystyle\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} 25 - x^2 - y^2 \, dy \, dx$

We would have to integrate terms like

$$\int \sqrt{25-x^2} \, dx, \quad \int x^2 \sqrt{25-x^2} \, dx, \quad \int (25-x^2)^{3/2} \, dx.$$

Too hard! Use polar coordinates, then

$$R = \{(r, \theta) : 0 \le r \le 5, \; 0 \le \theta \le 2\pi\}$$

$$f(x, y) = 25 - x^2 - y^2 = 25 - (x^2 + y^2) = 25 - r^2$$

Then $V = \displaystyle\int_0^{2\pi} \int_0^5 (25 - r^2) . r \, dr \, d\theta$

$$= \int_0^{2\pi} \left[\frac{25}{2} r^2 - \frac{1}{4} r^4 \right]_0^5 d\theta = \int_0^{2\pi} \frac{625}{4} d\theta$$

$$= \frac{625\pi}{2}.$$

*DON'T FORGET THE EXTRA 'r'!

Change of coordinates in triple integrals

This is very similar to the case of double integrals – in an integral of the form

$$\iiint\limits_R f(x, y, z)\, dx\, dy\, dz,$$

where R is a solid in $\mathbb{R}^3$, we change the coordinates x, y, z to new coordinates u, v, w using a transformation $\mathbf{g}$:

$$(x, y, z) = \mathbf{g}(u, v, w),$$

$$dx\, dy\, dz = \left| \det \left(\frac{\partial \mathbf{g}(u, v, w)}{\partial (u, v, w)} \right) \right| du\, dv\, dw,$$

$$S = \mathbf{g}^{-1}(R),$$

where S is the pre-image of R under $\mathbf{g}$, that is, the domain of the new coordinates u, v, w.

As before, the matrix $\left(\dfrac{\partial \mathbf{g}(u,v,w)}{\partial(u,v,w)}\right)$ is the *Jacobian matrix* of the transformation $\mathbf{g}(u,v,w)$ at (u,v,w), that is, if

$$\mathbf{g}(u,v,w) = (g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)),$$

then

$$\left(\frac{\partial \mathbf{g}(u,v,w)}{\partial(u,v,w)}\right) = \begin{bmatrix} \dfrac{\partial g_1(u,v,w)}{\partial u} & \dfrac{\partial g_1(u,v,w)}{\partial v} & \dfrac{\partial g_1(u,v,w)}{\partial w} \\ \dfrac{\partial g_2(u,v,w)}{\partial u} & \dfrac{\partial g_2(u,v,w)}{\partial v} & \dfrac{\partial g_2(u,v,w)}{\partial w} \\ \dfrac{\partial g_3(u,v,w)}{\partial u} & \dfrac{\partial g_3(u,v,w)}{\partial v} & \dfrac{\partial g_3(u,v,w)}{\partial w} \end{bmatrix}.$$

Its determinant $J(u,v,w)$ is called the *Jacobian* of $\mathbf{g}(u,v,w)$ at (u,v,w).

Then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_S f(\mathbf{g}(u,v,w)) \left|\det\left(\frac{\mathbf{g}(u,v,w)}{(u,v,w)}\right)\right| du\,dv\,dw,$$

where S is just a description of the domain of integration in terms of u, v, w instead of x, y, z.

Note that the $|\cdot|$ in the change of variable formula means absolute value, and not the determinant – the Jacobian is already defined as a determinant!

We will look at two special cases of change of variable in triple integrals which deal with two special types of 3-dimensional coordinate systems.

Cylindrical coordinates

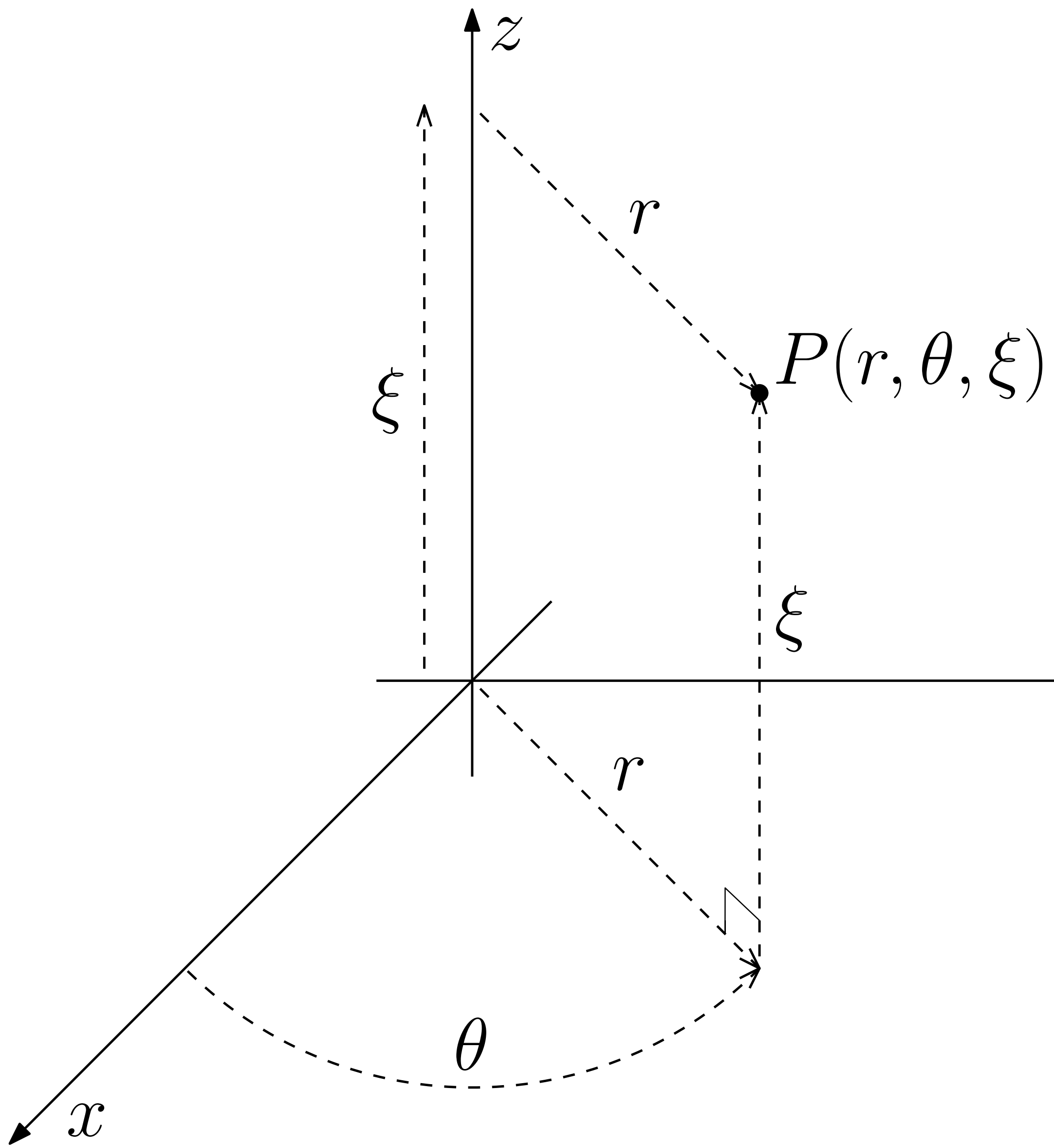
Everything we learned about polar coordinates we can apply to *cylindrical coordinates*, since cylinders are just discs with an added third dimension.

The cylindrical coordinates (r, θ, ξ) of a point P in $\mathbb{R}^3$ are obtained by representing the x and y coordinates in polar coordinates and letting the z coordinate be the z coordinate of the Cartesian coordinate system. Then

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad z = \xi,$$

$$r = \sqrt{x^2 + y^2} \quad , \quad 0 \leq \theta \leq 2\pi.$$

So we added a third Cartesian dimension $\xi = z$ and kept the same conventions for r and θ as for r and θ in polar coordinates.


z
r
ξ
P(r, θ, ξ)
ξ
y
r
θ
x

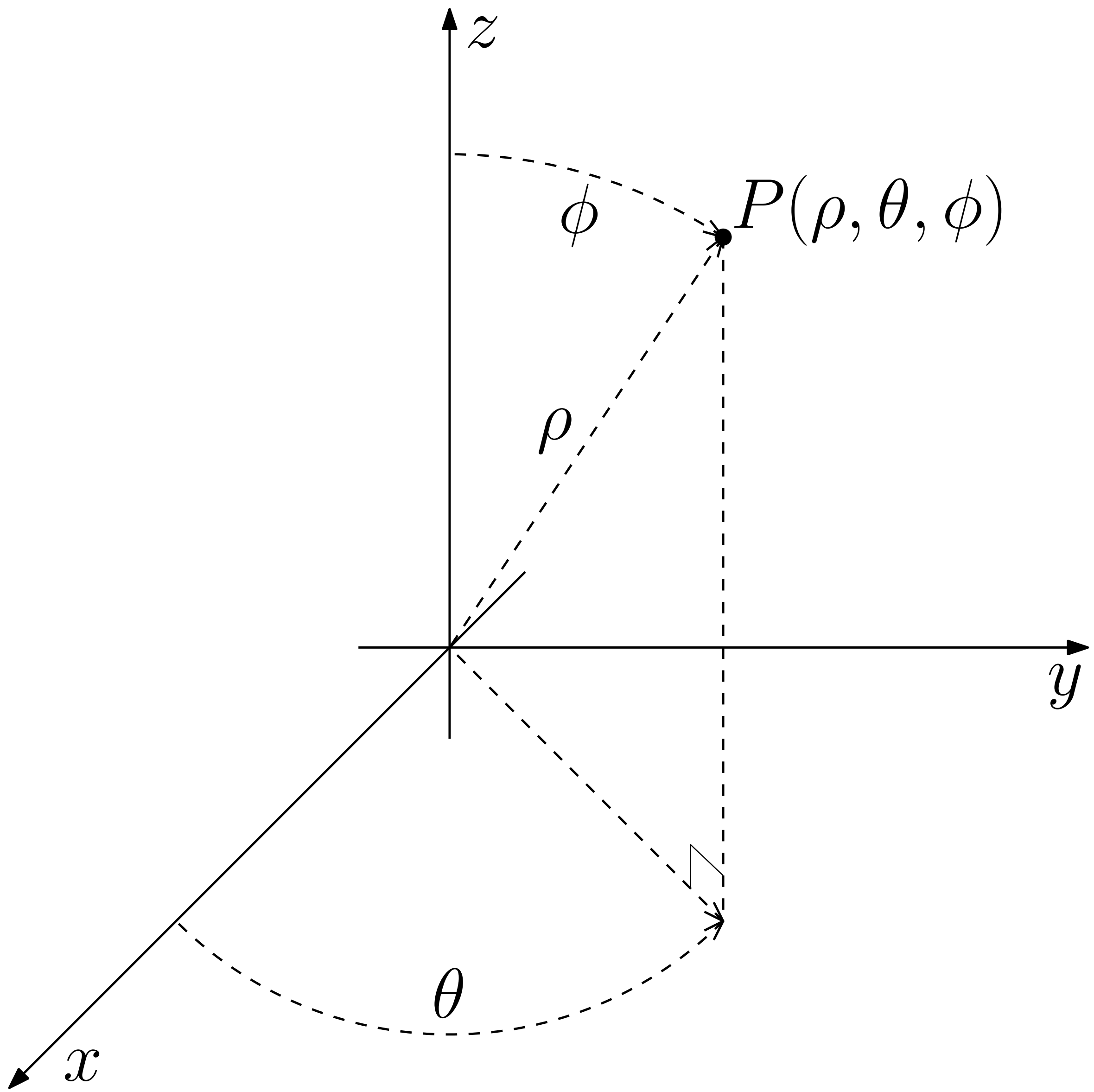
Spherical coordinates

While the cylindrical coordinate system is just the polar coordinate system with an added third Cartesian dimension, the spherical coordinate system requires transformation of all three Cartesian coordinates, but there are several similarities to the polar coordinate system.

As we have seen a disc can be described using one radius and one angle; to get the third dimension we just add another angle.

This time ρ is the distance from the origin, and θ is still the angle from x-axis in a plane parallel to the xy-plane. Our third coordinate ϕ is the angle down from the z-axis.

We can think about θ being the *longitude* and ϕ the *latitude* of a point P in $\mathbb{R}^3$.


z
ϕ
$P(\rho, \theta, \phi)$
ρ
y
θ
x

The relationship between Cartesian and spherical coordinates can easily be deduced by using basic trigonometry and Pythagoras' theorem:

$$x = \rho \cos\theta \sin\phi,$$

$$y = \rho \sin\theta \sin\phi,$$

$$z = \rho \cos\phi,$$

where

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

$$0 \leq \theta \leq 2\pi,$$

$$0 \leq \phi \leq \pi.$$

Example

The following tables list the Cartesian coordinates and corresponding cylindrical and spherical coordinates of a few points in space.

Cartesian	Cylindrical
$(1,0,0)$	$(1,0,0)$
$(-1,0,0)$	$(1,\pi,0)$
$(0,2,3)$	$\left(2,\frac{\pi}{2},3\right)$
$(1,1,2)$	$\left(\sqrt{2},\frac{\pi}{4},2\right)$
$(1,-1,2)$	$\left(\sqrt{2},\frac{7\pi}{4},2\right)$

Cartesian	Spherical
$(1,0,0)$	$\left(1,0,\frac{\pi}{2}\right)$
$(0,0,-1)$	$(1,0,\pi)$
$\left(0,1,\sqrt{6}\right)$	$\left(2\sqrt{2},\frac{\pi}{4},\frac{\pi}{6}\right)$
$\left(1,1,\sqrt{2}\right)$	$\left(2,\frac{\pi}{4},\frac{\pi}{4}\right)$
$\left(1,-1,-\sqrt{2}\right)$	$\left(2,\frac{7\pi}{4},\frac{3\pi}{4}\right)$

Triple integrals in cylindrical coordinates

For cylindrical coordinates we use the transformation function:

$$\mathbf{g}(r,\theta,\xi) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ \xi \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The Jacobian of the transformation $\mathbf{g}$ is:

$$\det\left(\frac{\partial \mathbf{g}(r,\theta,\xi)}{\partial(r,\theta,\xi)}\right) = \det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r > 0.$$

Then, for $S = \mathbf{g}^{-1}(R)$,

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz = \iiint\limits_S f(r\cos\theta, r\sin\theta, \xi)\,r\,dr\,d\theta\,d\xi.$$

Triple integrals in spherical coordinates

For spherical coordinates we use the transformation function:

$$\mathbf{g}(\rho,\theta,\phi)=\begin{bmatrix}\rho\cos\theta\sin\phi\\ \rho\sin\theta\sin\phi\\ \rho\cos\phi\end{bmatrix}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}.$$

The Jacobian of the transformation $\mathbf{g}$ is:

$$\det\left(\frac{\partial\mathbf{g}(\rho,\theta,\phi)}{\partial(\rho,\theta,\phi)}\right)=\det\begin{bmatrix}\cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi\\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi\\ \cos\phi & 0 & -\rho\sin\phi\end{bmatrix}$$

$$=-\rho^2\sin\phi.$$

Since $0 \le \phi \le \pi$, $\sin\phi$ is non-negative and we have

$$\left|\det\left(\frac{\partial \mathbf{g}(\rho,\theta,\phi)}{\partial(\rho,\theta,\phi)}\right)\right| = \rho^2 \sin\phi.$$

Then, for $S = \mathbf{g}^{-1}(R)$,

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint\limits_S f(\rho\cos\theta\sin\phi,\, \rho\sin\theta\sin\phi,\, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

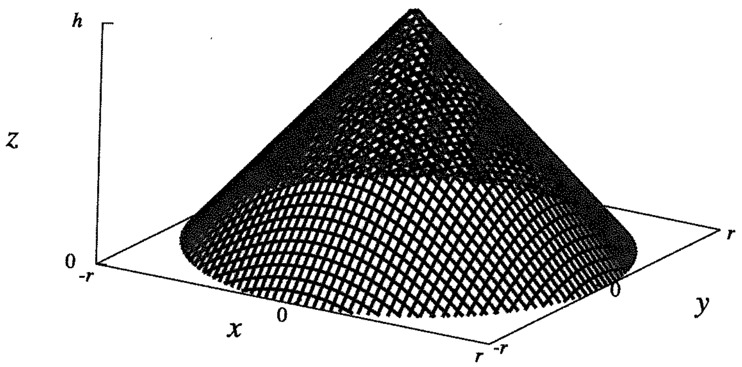
Example set 1 – week 10

MATH1011 Class Examples Week 10

Example set 1

① Determine the volume of a right circular cone of base radius a and vertical height h, which is described in Cartesian coordinates by the inequalities

$$x^2 + y^2 \leq a^2\left(1 - \frac{z}{h}\right)^2, \quad 0 \leq z \leq h.$$



In cylindrical coordinates we have

$$0 \leq r \leq a\left(1 - \frac{z}{h}\right), \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq h$$

Then $V = \int_0^h \int_0^{2\pi} \int_0^{a(1-z/h)} r \, dr \, d\theta \, dz$

$$= \int_0^h \int_0^{2\pi} \left[\tfrac{1}{2} r^2\right]_0^{a(1-z/h)} d\theta \, dz$$

$$= \int_0^h \int_0^{2\pi} \frac{1}{2} \cdot a^2 \cdot \left(1 - \frac{z}{h}\right)^2 . \, d\theta \, dz$$

$$= \int_0^h \int_0^{2\pi} \frac{1}{2} a^2 \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) d\theta \, dz$$

$$= \int_0^h \left[\frac{1}{2} a^2 \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) . \theta \right]_0^{2\pi} dz$$

$$= \pi a^2 \int_0^h 1 - \frac{2}{h} z + \frac{1}{h^2} z^2 \, dz$$

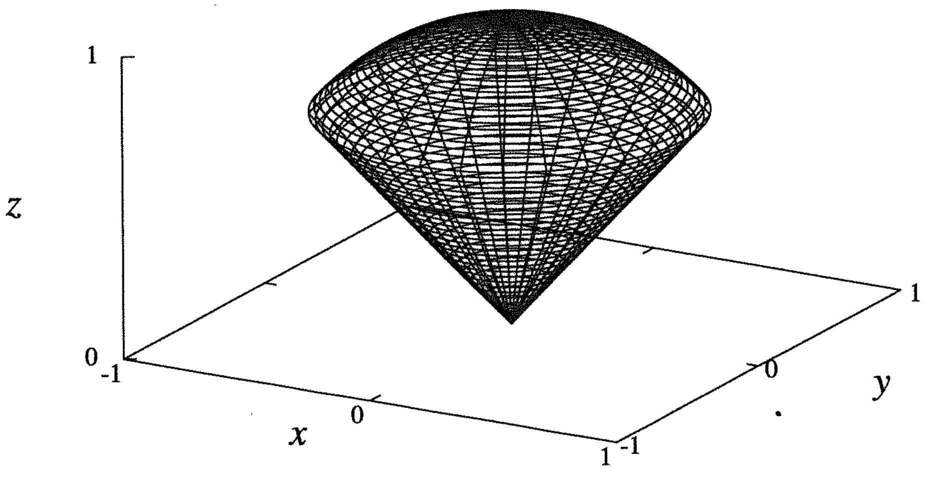
$$= \pi a^2 \left[z - \frac{1}{h} z^2 + \frac{1}{3h^2} z^3 \right]_0^h$$

$$= \pi a^2 \left[h - h + \frac{1}{3} h \right]$$

$$= \frac{1}{3} \pi a^2 h.$$

(2) Evaluate the volume of an ice cream cone described in spherical coordinates by

$$0 \leq \rho \leq R, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/4$$



$$V = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/4}\int_0^{2\pi} \left[\frac{1}{3}\rho^3 \sin\phi\right]_0^R d\theta \, d\phi$$

$$= \int_0^{\pi/4}\int_0^{2\pi} \frac{1}{3} R^3 \sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi/4} \left[\frac{1}{3} R^3 \sin\phi \cdot \theta\right]_0^{2\pi} d\phi$$

$$= \frac{2\pi}{3} R^3 \int_0^{\pi/4} \sin\phi \, d\phi$$

$$= \frac{2\pi}{3} R^3 \left[-\cos\phi\right]_0^{\pi/4}$$

$$= \frac{2\pi}{3} R^3 \left[-\cos\left(\frac{\pi}{4}\right) + \cos(0)\right]$$

$$= \frac{2\pi}{3} R^3 \left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$= \frac{2\pi}{3} R^3 \left(1 - \frac{1}{\sqrt{2}}\right)$$