Path and surface integrals

Recall that we can calculate the *length of a curve* $C = \{(x, f(x)) : a \leq x \leq b\}$ in $\mathbb{R}^2$ via an integral of the form

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

In applications it is often necessary to find lengths of curves on more complicated surfaces.

For example, how could we determine the length of a path taken by a climber as they scaled the side of a mountain?

Unless the mountainside is particularly simple, the techniques we know about so far are not sufficient to enable us to calculate the length of the route taken.

Moreover, can we evaluate the surface area of the mountain itself?

Such questions require knowledge of the methods of *path integrals* and *surface integrals*.

Before we begin to tackle these new types of integrals it is important to revise our knowledge of parametric representations of curves and surfaces.

It will turn out that determining suitable parametric forms is the starting point of path and surface integrals, and the ability to find such parametrisations is one that needs to be mastered.

Parametric forms of paths

Given some curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ an object travels along in the time $t \in [a, b]$, its location can be determined by finding the position vector $\mathbf{r}(t)$.

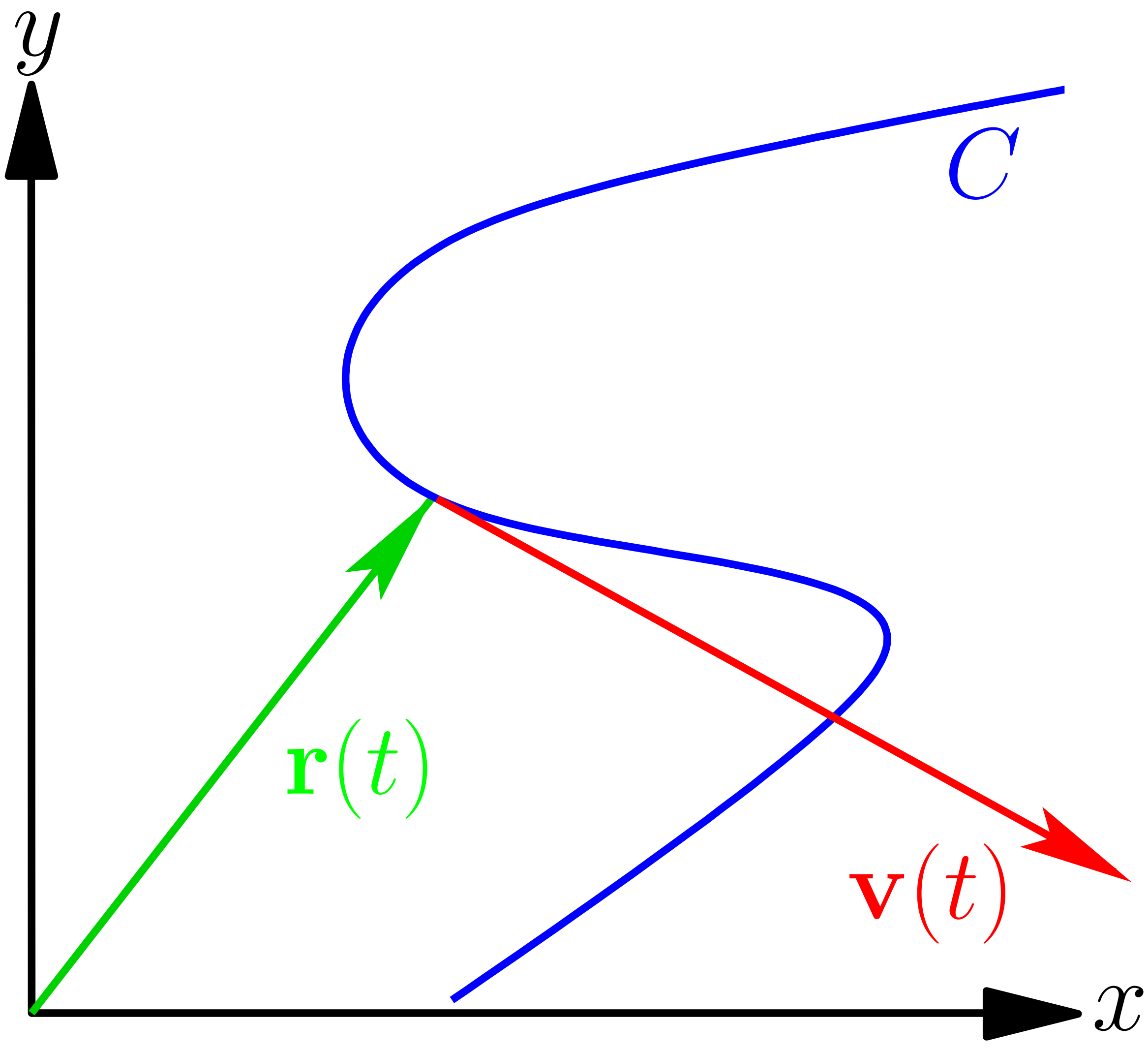
We can determine the velocity $\mathbf{v}(t)$ of the object by finding the tangent to the curve. Then we have the "position" vector:

$$\mathbf{r}(t) = (x, y) \text{ in } \mathbb{R}^2 \quad , \quad \mathbf{r}(t) = (x, y, z) \text{ in } \mathbb{R}^3,$$

and the "velocity" vector:

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t) = \dot{\mathbf{r}}(t),$$

where we have used the usual notation of a a dot instead of a dash to represent derivatives with respect to time. By varying t from a to b we obtain a function $\mathbf{r}(t)$ that is called the *parametric form* of the curve. We denote by C the curve in its entirety, that is, $C = \{\mathbf{r}(t) | t \in [a, b]\}$.


y
C
$\mathbf{r}(t)$
$\mathbf{v}(t)$
x

Example

To find a parametric form of the straight line from $(1, 2)$ to $(3, 6)$, we try to find functions $x(t)$ and $y(t)$ that together form the components of the parametric form $\mathbf{r}(t) = (x(t), y(t))$.

Both of them will be linear and therefore of the form:

$$x(t) = x_0 + m_x t \quad , \quad y(t) = y_0 + m_y t.$$

We can choose any closed t interval over which our parametrisation is to be applied – suppose we take $0 \leq t \leq 1$ as the domain of the function. Then $t = 0$ corresponds to the starting point $(1, 2)$, which determines the values $x(0) = x_0 = 1$ and $y(0) = y_0 = 2$. And with $t = 1$ corresponding to the end point $(3, 6)$ we must have and therefore $x(1) = 3$ and $y(1) = 6$, implying $m_x = 2$ and $m_y = 4$.

Therefore a parametric form of the line is:

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 + 2t \\ 2 + 4t \end{bmatrix}, \quad t \in [0, 1].$$

This simple example highlights several important points:

1. A parametric form of a curve is a vector-valued function $\mathbf{r}(t)$.

2. We need to specify the domain on which $\mathbf{r}(t)$ is defined: $t \in [a, b]$.

3. The range of $\mathbf{r}(t)$ includes both the starting and end points: $\mathbf{r}(a)$ is the starting point and $\mathbf{r}(b)$ is the end point.

Example

To find a parametric form of the circle $x^2 + y^2 = 1$, in Cartesian coordinates we would need to express the circle in two different segments

$$y = \sqrt{1 - x^2}, \quad y = -\sqrt{1 - x^2}.$$

If we further require that a particle moves once around the circle starting and ending at $(1, 0)$, we can write it as the union of these rather messy parametric forms:

$$\mathbf{r}(t) = \begin{bmatrix} 1 - t \\ \\ \sqrt{1 - (1 - t)^2} \end{bmatrix}, \quad t \in [0, 2],$$

$$\mathbf{r}(t) = \begin{bmatrix} t - 1 \\ \\ -\sqrt{1 - (t - 1)^2} \end{bmatrix}, \quad t \in [0, 2].$$

On the other hand, in polar coordinates the circle can be expressed in a much easier single form:

$$\mathbf{r}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad t \in [0, 2\pi].$$

So, for a curve C given by an explicit equation $y = f(x)$ on the domain $x \in \mathcal{D}$, if we let (x_0, y_0) be the starting point and (x_1, y_1) the end point of the curve, then a parametric form $\mathbf{r}(t)$ of C can be found by choosing $t = x$, where $t \in [x_0, x_1]$. Then

$$\mathbf{r}(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix}, \quad t \in [x_0, x_1].$$

However, remember that parametric forms are **not unique** representations of curves, and the simplest parameterisation may not necessary be the most appropriate one.

Example set 2 – week 10

Example set 2

① Find a parametric form for the circle of radius 3 centred at $(x, y) = (3, 2)$.

This has equation

$$(x-3)^2 + (y-2)^2 = 9$$

So if we let

$$x - 3 = 3\sin t$$
$$y - 2 = 3\cos t$$

then $9\sin^2 t + 9\cos^2 t = 9(\sin^2 t + \cos^2 t)$

$$= 9$$

So one possible parameterisation is

$$x = 3 + 3\sin t, \quad y = 2 + 3\cos t$$

A particle at $t=0$ starts at $(x, y) = (3, 5)$ and travels around the circle anti-clockwise for $0 \leq t \leq 2\pi$.

② Find a parametric form for the ellipse

$$9x^2 + 25y^2 = 225.$$

We write this as $\frac{9}{225} x^2 + \frac{25}{225} y^2 = 1$

so $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

or $\left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 1$

$\therefore$ Let $\dfrac{x}{5} = \sin t$, $\dfrac{y}{3} = \cos t$

Then a suitable parameterization is

$$x = 5\sin t, \quad y = 3\cos t, \quad t \in [0, 2\pi]$$

(3) Find a parametric representation of $xy = 4$.

Since $y = \dfrac{4}{x}$ we could take

$$x = t, \quad y = \frac{4}{t}, \quad t > 0 \text{ or } t < 0.$$

(4) Find a parametric representation of $xy - x - 4y + 2 = 0$

We have $y(x-4) = x - 2 \Rightarrow y = \dfrac{x-2}{x-4}$

So we could take $x - 2 = t \Rightarrow x = t + 2$
and then $y = \dfrac{t}{t+2-4} = \dfrac{t}{t-2}$

$$\Rightarrow x = t + 2, \quad y = \frac{t}{t-2}.$$

Length of curves

The following theorem explains how parametric forms can be used to determine *lengths of curves*:

If C is given in parametric form by $\{\mathbf{r}(t) | a \leq t \leq b\}$, and the vector function $\mathbf{r}(t)$ is differentiable in (a, b) and continuous on $[a, b]$, then the length $L(C)$ of C is given by

$$L(C) = \int_a^b \left| \frac{d\mathbf{r}}{dt} \right| dt = \int_a^b \sqrt{\dot{r_1}^2 + \ldots + \dot{r_p}^2}\, dt,$$

where $\mathbf{r}(t) = (r_1(t), \cdots, r_p(t))$, $\dot{r}_i = \dfrac{dr_i}{dt}$, and $p = 2$ or 3 according to the ambient space being $\mathbb{R}^2$ or $\mathbb{R}^3$.

See the Unit Reader for a derivation of this result using Riemann sums.

Example

To determine the length of the spiral S given by the parametrisation $\mathbf{r} : [0, 2\pi] \mapsto \mathbb{R}^3$, where $\mathbf{r}(t) = (\cos t,\ \sin t,\ t)$, we compute

$$\dot{\mathbf{r}}(t) = (-\sin t,\ \cos t, 1),$$

and therefore

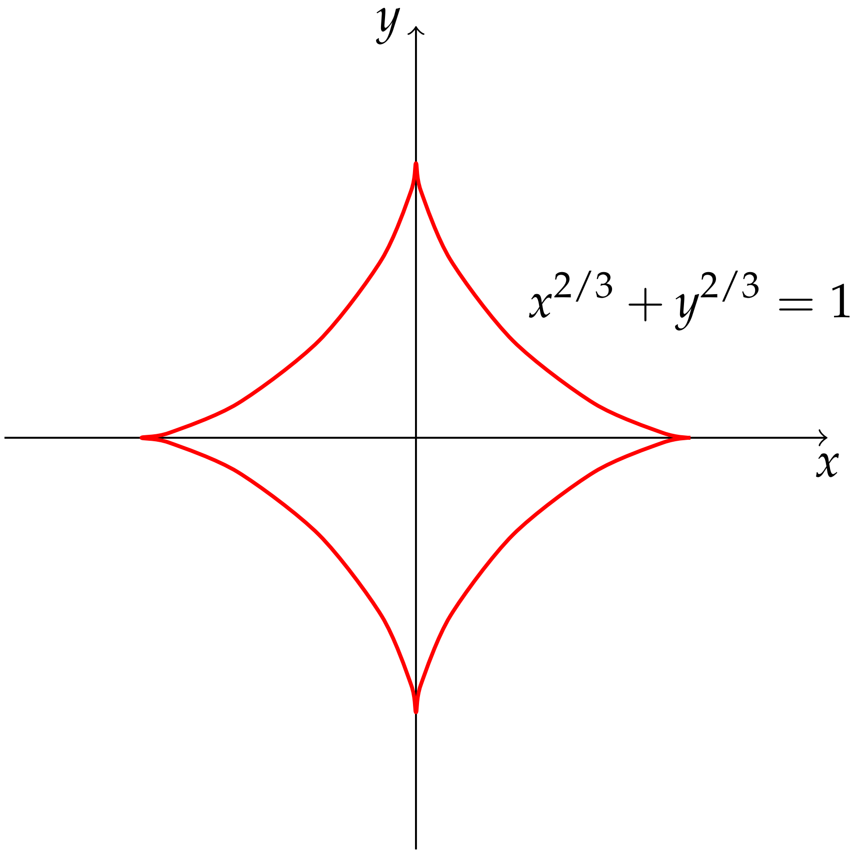
$$|\dot{\mathbf{r}}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2},$$

for all t. Hence the length of the spiral is

$$L(S) = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}[t]_0^{2\pi} = 2\sqrt{2}\pi.$$

Example

Find the length of the part of the astroid $x^{2/3}+y^{2/3} = 1$ which is contained in the second quadrant ($x \leq 0$, $y \geq 0$).



Note that $x^{2/3} + y^{2/3}$ may be written $\left(x^{1/3}\right)^2 + \left(y^{1/3}\right)^2$, so that $x^{1/3}$ and $y^{1/3}$ are on a circle of radius 1, so we can use the parametrisation

$$\begin{bmatrix} x^{1/3} \\ y^{1/3} \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \Rightarrow \mathbf{r}(t) = \begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix}.$$

Since our curve is in the second quadrant, it goes from $(0, 1)$ to $(-1, 0)$ which corresponds to $\pi/2 \le t \le \pi$. Also,

$$\dot{\mathbf{r}}(t) = \begin{bmatrix} -3\cos^2 t \sin t \\ 3\sin^2 t \cos t \end{bmatrix},$$

so

$$|\dot{\mathbf{r}}| = \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)} = \sqrt{9\cos^2 t \sin^2 t} = -3\cos t \sin t.$$

Notice that the quantity $|\dot{\mathbf{r}}|$ must be **positive** and that for the range $\pi/2 \leq t \leq \pi$ it is the case that $\cos t \leq 0$ and $\sin t \geq 0$, so $\cos t \sin t \leq 0$.

Hence why we take the negative square root!

Then

$$\begin{aligned} L &= \int_{\pi/2}^{\pi} -3\cos t \sin t \, dt \\ &= -\frac{3}{2}\left[\sin^2 t\right]_{\pi/2}^{\pi} \\ &= -\frac{3}{2}(0-1) = \frac{3}{2}. \end{aligned}$$

Example set 3 – week 10

Example set 3

① Find the length of the parametric curve

$$x = t^2, \quad y = t^3, \quad t \in [1, 2]$$

Here we have

$$\bar{r}(t) = (t^2, t^3)$$

so $\quad \bar{r}'(t) = (2t, 3t^2)$

and $\quad |\bar{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2}$

$$= \sqrt{4t^2 + 9t^4}$$

$$= \sqrt{t^2(4 + 9t^2)}$$

$$= t\sqrt{4 + 9t^2}$$

Then $\quad L = \int_1^2 t\sqrt{4 + 9t^2}\, dt$

Change of variable

$u = 4 + 9t^2$

$\frac{du}{dt} = 18t$

$dt = \frac{du}{18t}$

$$= \int_{13}^{40} t \cdot \sqrt{u} \cdot \frac{du}{18t}$$

$$= \frac{1}{18} \int_{13}^{40} u^{1/2}\, du$$

Change of limits

$t = 1, \ u = 13$

$t = 2, \ u = 40$

$$= \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_{13}^{40}$$

$$= \frac{1}{27} \left[40^{3/2} - 13^{3/2} \right] = \frac{1}{27} \left(80\sqrt{10} - 13\sqrt{13} \right)$$

(2) Find the length of the parametric curve

$$\bar{r}(t) = \left(3t, \sqrt{3}t^2, \tfrac{2}{3}t^3\right)$$

for $t \in [0,1]$.

$$\bar{r}'(t) = \left(3, 2\sqrt{3}t, 2t^2\right)$$

$$|\bar{r}'(t)| = \sqrt{3^2 + (2\sqrt{3}t)^2 + (2t^2)^2}$$

$$= \sqrt{9 + 12t^2 + 4t^4}$$

$$= \sqrt{(2t^2+3)^2}$$

$$= 2t^2 + 3$$

$$L = \int_0^1 2t^2 + 3\,dt$$

$$= \left[\tfrac{2}{3}t^3 + 3t\right]_0^1$$

$$= \frac{2}{3} + 3$$

$$= \frac{11}{3}$$

Path integrals of a function

Let $f(x, y)$ or $f(x, y, z)$ be a continuous function defined on a smooth curve C parameterised by $\mathbf{r}(t)$ for $t \in [a, b]$.

This means that f can be evaluated for all $t \in [a, b]$ and we require the function to be continuous in t. Moreover we require that $\dot{\mathbf{r}}$ exists and is continuous.

We can now define *path integrals*:

We define the integral of a function $f(x, y)$ or $f(x, y, z)$ over a path $C = \{\mathbf{r}(t) | a \leq t \leq b\}$ to be

$$\int_C f \, ds = \int_C f(\mathbf{r}(t)) \, |\dot{\mathbf{r}}(t)| \, dt.$$

We can think of this integral (sometimes also called a *line integral*) as an integration of f along a special path.

We need to evaluate f on the path, which we indicate by $f(x(t), y(t))$ or $f(x(t), y(t), z(t))$.

Example

Evaluate the path integral $\displaystyle\int_C x + y + z\, ds$ where C has the parametric equation

$$\mathbf{r}(t) = (\cos t, \sin t, t),\ 0 \le t \le \pi.$$

First we calculate the tangent vector:

$$\dot{\mathbf{r}}(t) = (-\sin t, \cos t, 1) \quad \Rightarrow \quad |\dot{\mathbf{r}}(t)| = \sqrt{2}.$$

Now our function is $f(x, y, z) = x + y + z$, and to evaluate it on the path we substitute the components of $\mathbf{r}$:

$$f(x(t), y(t), z(t)) = \cos t + \sin t + t,$$

and hence

$$\begin{aligned}
\int_C x + y + z \, ds &= \int_0^\pi (\cos t + \sin t + t)\sqrt{2}\, dt \\
&= \sqrt{2}\left[\sin t - \cos t + \frac{t^2}{2}\right]_0^\pi \\
&= \sqrt{2}\left(2 + \frac{\pi^2}{2}\right).
\end{aligned}$$

Example set 4 – week 10

Example set 4

① Evaluate the path integral

$$\int_C xy^2 \, ds$$

where C is the quarter-circle defined by

$$x = 4\cos t, \quad y = 4\sin t, \quad t \in [0, \pi/2]$$

Here we have

$$f(x,y) = xy^2 \Rightarrow f(4\cos t, 4\sin t)$$

$$= (4\cos t)(4\sin t)^2$$

$$= 64\sin^2 t \cos t$$

and $\bar{r}(t) = (4\cos t, 4\sin t)$

$$\Rightarrow \bar{r}'(t) = (-4\sin t, 4\cos t)$$

$$\Rightarrow |\bar{r}'(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2}$$

$$= \sqrt{16(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{16} = 4$$

So $\displaystyle\int_C xy^2 \, ds = 256 \int_0^{\pi/2} \sin^2 t \cos t \, dt$

Now, $\int_0^{\pi/2} \sin^2 t \cos t \, dt$

Change of variable

$u = \sin t$

$\frac{du}{dt} = \cos t$

$dt = \frac{du}{\cos t}$

$= \int_0^1 u^2 . \cos t . \frac{du}{\cos t}$

$= \int_0^1 u^2 \, du$

Change of limits

$t = 0, \; u = 0$

$t = \pi/2, \; u = 1$

$= \left[\frac{1}{3} u^3 \right]_0^1$

$= \frac{1}{3}$

$\therefore \int_C x y^2 \, ds = \frac{256}{3}$

(2) Compute $\int_C z + y^2 ds$ where C is the line segment from $(3, 4, 0)$ to $(1, 4, 2)$.

A suitable parameterization of C is

$\bar{r}(t) = (3 - 2t, \; 4, \; 2t) \quad , t \in [0, 1]$

Note that $\bar{r}(0) = (3, 4, 0)$
$\bar{r}(1) = (1, 4, 2)$.

Then $\bar{r}'(t) = (-2, 0, 2)$

$|\bar{r}'(t)| = 2\sqrt{2}$

$$\Rightarrow \int_C z + y^2 \, ds = \int_0^1 (2t + 4^2) \cdot 2\sqrt{2} \, dt$$

$$= \int_0^1 4\sqrt{2}\,t + 32\sqrt{2} \, dt$$

$$= \left[2\sqrt{2}\,t^2 + 32\sqrt{2}\,t \right]_0^1$$

$$= 2\sqrt{2} + 32\sqrt{2}$$

$$= 34\sqrt{2}$$

(3) Evaluate $\int_C 3x \, ds$ where C is the curve $y = x^2$ from $(0,0)$ to $(3,9)$

Take $\bar{r}(t) = (t, t^2)$, $t \in [0, 3]$

$\bar{r}'(t) = (1, 2t) \Rightarrow |\bar{r}'(t)| = \sqrt{1 + 4t^2}$

$$\therefore \int_C 3x \, ds = \int_0^3 3t\sqrt{1 + 4t^2} \, dt$$

$$= \frac{3}{8} \int_1^{37} u^{1/2} \, du \quad \text{[Check this!]}$$

$$= \frac{1}{4}\left(37^{3/2} - 1\right).$$

THE UNIVERSITY OF WESTERN AUSTRALIA

School of Mathematics and Statistics

MATH1011 MULTIVARIABLE CALCULUS
LECTURE NOTES WEEK 11

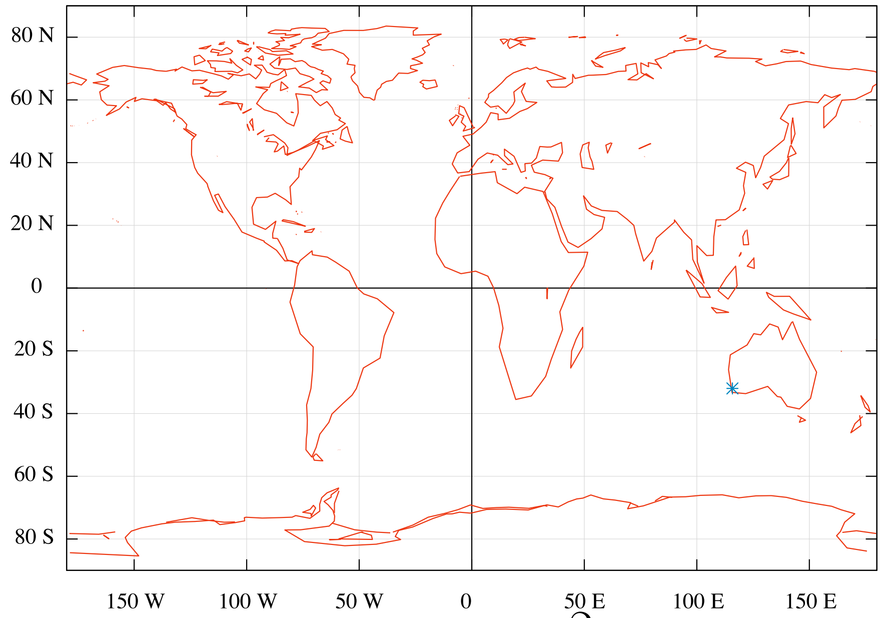
Parametric forms of surfaces

While we only need one parameter to describe a path in space, a surface is specified by an equation involving two parameters. As a familiar example we might want to think about the coordinates we use to describe our location on earth:

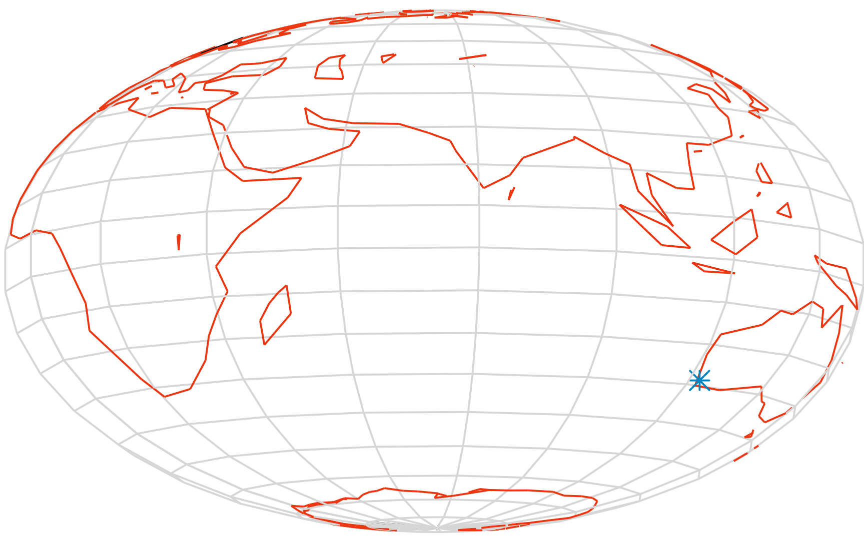
$$(x, y, z) = f(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v).$$

This function gives a point on a sphere of unit radius, where u represents *longitude*, v represents *latitude*.

This is obtained using spherical coordinates by setting $\rho = 1$, $u = \theta$ and $v = \phi$.



$\longmapsto$



$$(u, v) \in \mathbb{R}^2 \qquad \longmapsto \qquad (x, y, z) \in \mathbb{R}^3$$

Using the method we have learned for paths, we can define a position vector and its associated tangent vectors:

$$(x, y, z) = \mathbf{S}(u, v),$$

$$\mathbf{S}_u(u, v) = \frac{\partial \mathbf{S}}{\partial u}(u, v),$$

$$\mathbf{S}_v(u, v) = \frac{\partial \mathbf{S}}{\partial v}(u, v).$$

Since the surface S depends on two parameters, we obviously have two tangent vectors at each point (u, v).

We remember that it is often useful to work with the vector that is normal to the surface:

$$\mathbf{N} = \mathbf{S}_u \times \mathbf{S}_v.$$

In general this vector is a function of u and v.

Example

To find a parametric form of the surface S given by $z = x^2 - y^3$, $0 \leq x \leq 1$ and $0 \leq y \leq 4$, we see that z depends only on two variables (x and y) and therefore we can immediately conclude:

$$x = u \quad , \quad y = v \quad , \quad z = u^2 - v^3.$$

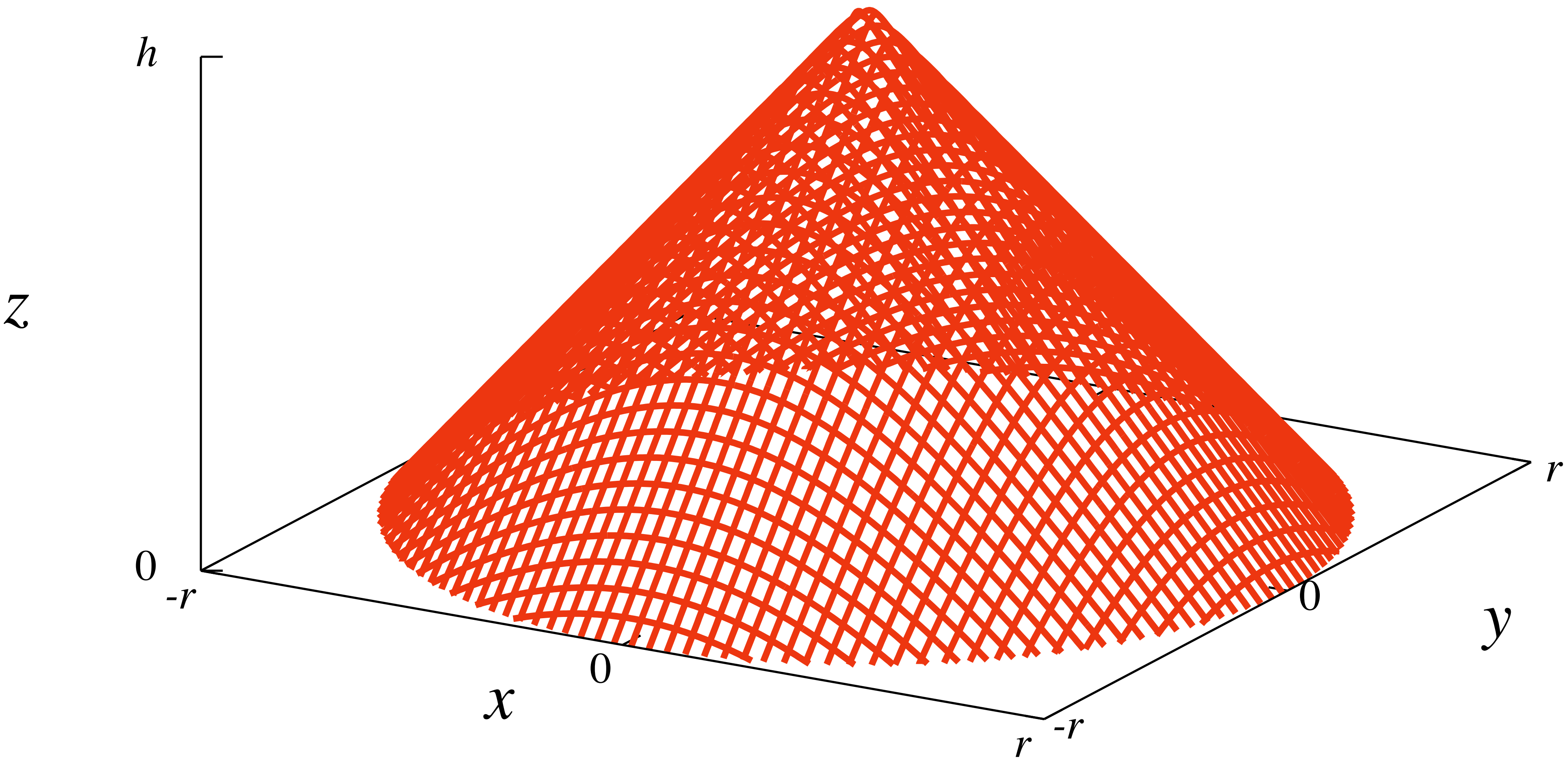
Therefore

$$\mathbf{S}(u, v) = (u, v, u^2 - v^3) \quad , \quad 0 \leq u \leq 1 \quad , \quad 0 \leq v \leq 4.$$

Example

Recall from last week that the Cartesian equations describing a cone with height h, basis radius a and apex at $(0, 0, 0)$ was

$$x^2 + y^2 = a^2 \left(1 - \frac{z}{h}\right)^2, \qquad 0 \leq z \leq h.$$


h
z
0
-r
0
x
r
-r
0
y
r

In cylindrical coordinates (with $x = r\cos\theta$ and $y = r\sin\theta$) we have

$$r = a\left(1 - \frac{z}{h}\right), \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq z \leq h.$$

As we can see the radius r depends on z but θ and z do not depend on any other variable.

This makes z and θ our favoured choice for u and v. We get:

$$x = a\left(1 - \frac{u}{h}\right)\cos v, \quad y = a\left(1 - \frac{u}{h}\right)\sin v, \quad z = u$$

$$\mathbf{S}(u, v) = \left(a\left(1 - \frac{u}{h}\right)\cos v, a\left(1 - \frac{u}{h}\right)\sin v, u\right),$$

$$0 \leq u \leq h, \quad 0 \leq v \leq 2\pi.$$

Example set 1 – week 11

MATH1011 Class Examples Week 11

Example set 1

① Consider a surface with parametric form

$$\bar{S}(u,v) = (2\cos u, v, 2\sin u)$$

- Find the Cartesian equation for this surface.

Here we have $x = 2\cos u$, $y = v$, $z = 2\sin u$ and we notice that

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$

So this is a circular cylinder of radius 2 whose axis is the y-axis.

② Find a parametric form for the elliptic paraboloid $z = x^2 + 2y^2$.

Setting $x = u$, $y = v$ we have simply

$$\bar{S}(u,v) = (u, v, u^2 + 2v^2).$$

③ Find a parametric form of the sphere

$$x^2 + y^2 + z^2 = a^2$$

Using spherical coordinates with $\rho = a$, $u = \theta$, $v = \phi$:

$$\bar{S}(u,v) = (a\cos u\sin v, a\sin u\sin v, a\cos v)$$

for $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi$.

Areas of surfaces

We will generalize the techniques of path integrals to show how we can evaluate *areas of surfaces* and double integrals of functions defined over surfaces which are embedded in $\mathbb{R}^3$.

Let $\mathbf{S}(u, v)$ be a differentiable (and hence continuous) surface in $\mathbb{R}^3$ (defined on some domain $D:\ a \leq u \leq b,\ c \leq v \leq d$). We denote

$$S = \{\mathbf{S}(u, v) |\ a \leq u \leq b,\ c \leq v \leq d\},$$

and we would like to determine the surface area of S.

At each point (u, v), the derivatives $\mathbf{S}_u$ and $\mathbf{S}_v$ exist and hence the **normal vector**

$$\mathbf{N}(u, v) = \mathbf{S}_u \times \mathbf{S}_v,$$

is well-defined.

Theorem

Let S be the surface given by a continuously differentiable parametrisation $S = \{\mathbf{S}(u, v) |\ (u, v) \in D\}$ for some region D in the u, v-plane.

Then the surface area of S, denoted by $\text{Area}(S)$, is equal to

$$\iint\limits_D \left| \frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v} \right| du\, dv = \iint\limits_D |\mathbf{N}(u, v)|\, du\, dv.$$

See the Unit Reader for a proof of this theorem using Riemann sums.

This means that we can find the area of a quite complicated surface S if it can be conveniently parameterised in terms of u and v.

Then, rather than trying to evaluate the area of $\mathbf{S}$ directly, we instead calculate the integral of

$$\left|\frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v}\right|,$$

over an appropriate region D.

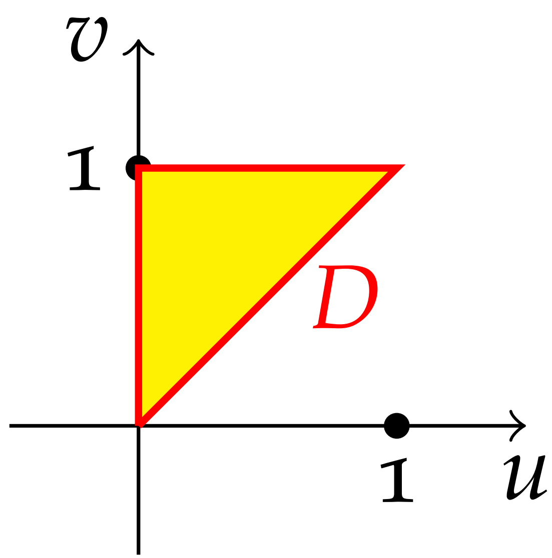
Example

To find the area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$, we use $u = x$ and $v = y$ to parametrise the surface:

$$\mathbf{S}(u,v) = (u, v, u + v^2),$$

and

$$D = \{(u,v) | 0 \leq v \leq 1, 0 \leq u \leq v\}.$$



Then the tangent vectors are $\mathbf{S}_u = (1, 0, 1)$ and $\mathbf{S}_v = (0, 1, 2v)$ and hence

$$\mathbf{S}_u \times \mathbf{S}_v = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2v \end{bmatrix} = (-1, -2v, 1),$$

which implies

$$|\mathbf{S}_u \times \mathbf{S}_v| = \sqrt{2 + 4v^2}.$$

Therefore we get

$$
\begin{aligned}
A &= \int_0^1 \int_0^v \sqrt{2+4v^2}\,du\,dv \\
&= \int_0^1 v\sqrt{2+4v^2}\,dv \\
&= \frac{1}{12}\left[(2+4v^2)^{3/2}\right]_0^1 \\
&= \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{6},
\end{aligned}
$$

where we have used the substitution $w = 2+4v^2$ to solve the last integral.

Example

Find the surface area of the *torus* (doughnut)

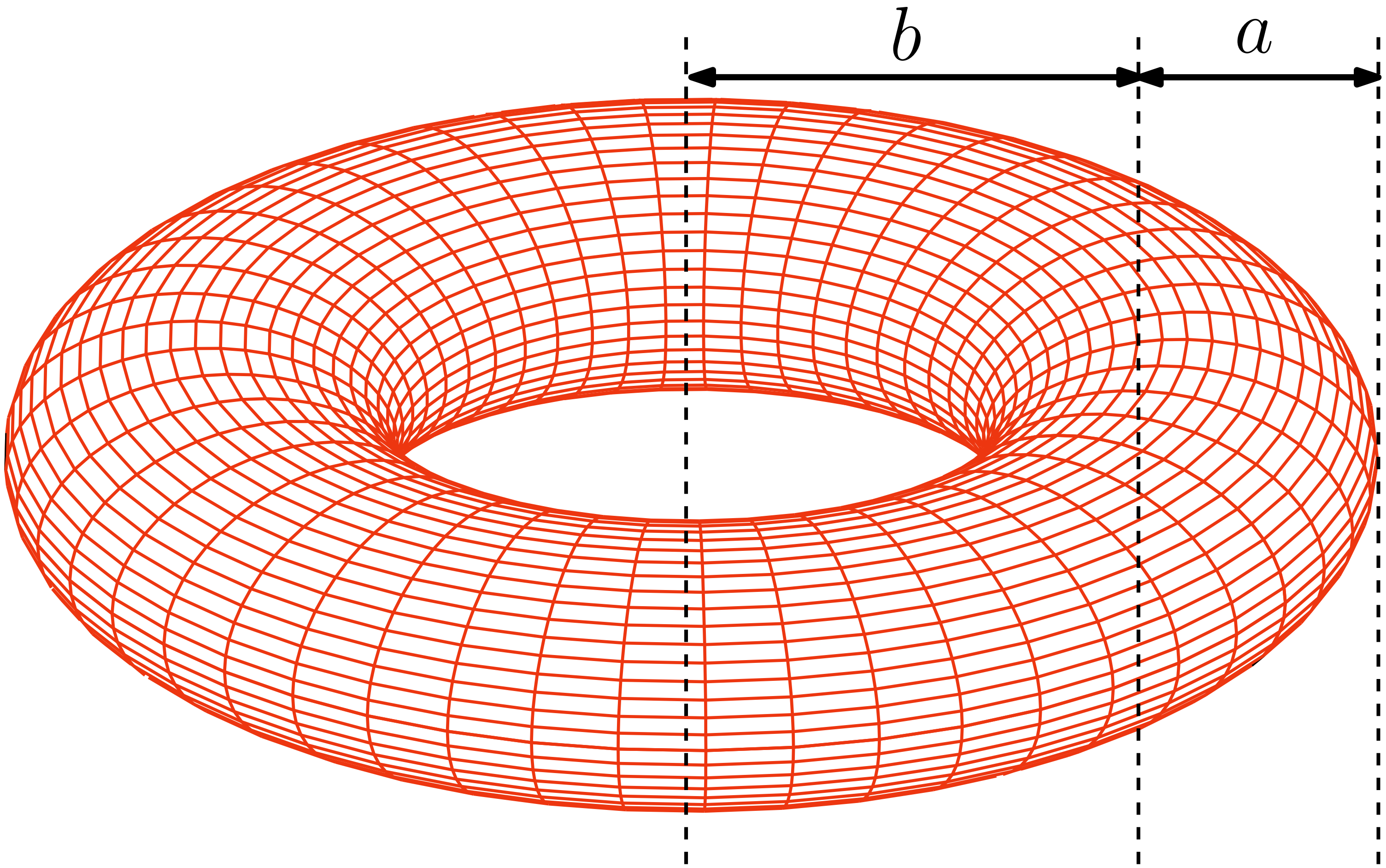
$$\mathbf{S}(\alpha,\theta) = \left((b + a\cos\alpha)\cos\theta, (b + a\cos\alpha)\sin\theta, a\sin\alpha\right),$$

where $0 \le \alpha \le 2\pi,\ 0 \le \theta \le 2\pi$.

Note that b is the radius of the circular centre of the torus and a is the radius of a vertical cross-section of the torus.

These two numbers are constant and $a < b$.

Our two parameters are θ, the angle on the torus around from $x = 0$, and α, the angle on this around from $z = 0$.


b
a

We get

$$\begin{aligned}\mathbf{S}_\alpha &= (-a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha)\\ &= a(-\sin\alpha\cos\theta, -\sin\alpha\sin\theta, \cos\alpha),\end{aligned}$$

and

$$\begin{aligned}\mathbf{S}_\theta &= (-(b+a\cos\alpha)\sin\theta, (b+a\cos\alpha)\cos\theta, 0)\\ &= (b+a\cos\alpha)(-\sin\theta, \cos\theta, 0).\end{aligned}$$

Then

$$\begin{aligned}\mathbf{N} = \mathbf{S}_\alpha \times \mathbf{S}_\theta &= a(b+a\cos\alpha)\det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin\alpha\cos\theta & -\sin\alpha\sin\theta & \cos\alpha\\ -\sin\theta & \cos\theta & 0\end{bmatrix}\\ &= -a(b+a\cos\alpha)(\cos\alpha\cos\theta, \cos\alpha\sin\theta, \sin\alpha),\end{aligned}$$

and so

$$\begin{aligned}|\mathbf{N}| &= a|b + a\cos\alpha|\sqrt{\cos^2\theta\cos^2\alpha + \sin^2\theta\cos^2\alpha + \sin^2\alpha} \\ &= a(b + a\cos\alpha).\end{aligned}$$

Note that $b - a \le b + a\cos\alpha \le b + a$, and so $0 < b + a\cos\alpha$.

We can now compute the surface area:

$$A = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\theta\,d\alpha = 2\pi a\left[b\alpha + a\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab.$$

Note that this is the product of the perimeters of two circles of radii a and b.

Example set 2 – week 11

Example set 2

① Verify the formula for the surface area of a sphere of radius a, which is $4\pi a^2$.

Here we have

$$\bar{S}(u,v) = (a\cos u \sin v,\ a \sin u \sin v,\ a\cos v)$$

where $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi$.

$$\frac{\partial \bar{S}}{\partial u} = (-a\sin u \sin v,\ a\cos u \sin v,\ 0)$$

$$\frac{\partial \bar{S}}{\partial v} = (a\cos u \cos v,\ a\sin u \cos v,\ -a\sin v)$$

$$\frac{\partial \bar{S}}{\partial u} \times \frac{\partial \bar{S}}{\partial v} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u \sin v & a\cos u \sin v & 0 \\ a\cos u \cos v & a\sin u \cos v & -a\sin v \end{bmatrix}$$

$$= \hat{i}\left[-a^2\cos u \sin^2 v\right] + \hat{j}\left[-a^2 \sin u \sin^2 v\right] + \hat{k}\left[-a^2\sin^2 u \sin v \cos v - a^2\cos^2 u \sin v \cos v\right]$$

$$= (-a^2\cos u \sin^2 v,\ -a^2\sin u \sin^2 v,\ -a^2 \sin v \cos v)$$

Then

$$\left|\frac{\partial \bar{S}}{\partial u} \times \frac{\partial \bar{S}}{\partial v}\right| = \sqrt{a^4\cos^2 u \sin^4 v + a^4 \sin^2 u \sin^4 v + a^4 \sin^2 v \cos^2 v}$$

$$= \sqrt{a^4 \sin^4 v + a^4 \sin^2 v \cos^2 v}$$

$$= \sqrt{a^4 \sin^2 v (\sin^2 v + \cos^2 v)}$$

$$= \sqrt{a^4 \sin^2 v}$$

$$= a^2 \sin v$$

Then $\text{Area}(S) = \iint_D a^2 \sin v \, du\, dv$

$$= \int_0^{2\pi} \int_0^{\pi} a^2 \sin v \, dv\, du$$

$$= \int_0^{2\pi} \left[-a^2 \cos v\right]_0^{\pi} du$$

$$= \int_0^{2\pi} 2a^2 \, du$$

$$= \left[2a^2 u\right]_0^{2\pi}$$

$$= 4\pi a^2$$

Surface integrals of a function

Let $f(x, y, z)$ be a continuous function defined on a smooth surface

$$S = \{\mathbf{S}(u, v) |\ (u, v) \in D\},$$

parametrised by a continuously differentiable parametrisation $\mathbf{S}$ for some region D in the u, v-plane.

This means that f can be evaluated for all $(u, v) \in D$ and we require the function to be continuous in both u and v.

These assumptions imply that the normal vector $\mathbf{N}$ exists and is continuous.

We can now define *surface integrals*.

We define the integral of a function $f(x, y, z)$ over a surface

$$S = \{\mathbf{S}(u, v) | (u, v) \in D\},$$

to be

$$\int_S f\, dS = \iint_D f(\mathbf{S}(u, v))\, |\mathbf{N}(u, v)|\, du\, dv,$$

where

$$\mathbf{N}(u, v) = \frac{\partial \mathbf{S}}{\partial u}(u, v) \times \frac{\partial \mathbf{S}}{\partial v}(u, v).$$

This integral can be regarded as an integration of $f(x, y, z)$ on a special surface S.

We need to evaluate f on the surface, which we indicate by

$$f(x(u, v), y(u, v), z(u, v)).$$

Example

Evaluate the surface integral

$$\iint_S xy \, dS,$$

where S is the triangle with vertices $(1,0,0)$, $(0,2,0)$ and $(0,0,2)$.

The surface is contained in a plane so we can parametrise it as follows:

$$\mathbf{S}(u,v) = (1,0,0) + u(-1,2,0) + v(-1,0,2) = (1-u-v, 2u, 2v).$$

The points $(1,0,0)$, $(0,2,0)$, $(0,0,2)$ correspond respectively to $(u,v) = (0,0), (1,0), (0,1)$, so the domain is

$$D = \{(u,v) | 0 \leq u \leq 1, \ 0 \leq v \leq 1-u\}.$$

We compute

$$\left|\frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v}\right| = |(-1,2,0) \times (-1,0,2)| = |(4,2,2)| = 2\sqrt{6},$$

and

$$f(\mathbf{S}(u,v)) = (1-u-v)(2u).$$

So the surface integral is

$$\iint\limits_S xy \, dS = \int_0^1 \int_0^{1-u} (1-u-v)(2u)(2\sqrt{6}) \, dv \, du$$

$$= 2\sqrt{6} \int_0^1 2u \left[(1-u)v - \frac{v^2}{2}\right]_0^{1-u} du$$

$$= 2\sqrt{6}\int_0^1 u(1-u)^2\,du$$

$$= 2\sqrt{6}\left[\frac{u^2}{2} - \frac{2u^3}{3} + \frac{u^4}{4}\right]_0^1$$

$$= \frac{\sqrt{6}}{6} = \frac{1}{\sqrt{6}}.$$

Example set 3 – week 11

Example set 3

① Compute the surface integral

$$\iint_S x^2 \, dS$$

where S is the unit sphere $x^2+y^2+z^2=1$.

Here we have

$x = \cos u \sin v$, $y = \sin u \sin v$, $z = \cos v$

for $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi$.

Taking $\bar{S}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$

we find

$$\left| \frac{\partial \bar{S}}{\partial u} \times \frac{\partial \bar{S}}{\partial v} \right| = \sin v \quad \text{[see last example]}$$

and $f(\bar{S}) = (\cos u \sin v)^2$

$$= \cos^2 u \sin^2 v$$

So $$\iint_S x^2 \, dS = \int_0^{2\pi} \int_0^{\pi} \cos^2 u \sin^2 v . \sin v . dv \, du$$

$$= \int_0^{2\pi} \int_0^{\pi} \cos^2 u . \sin^3 v \, dv \, du .$$

$$= \int_0^{2\pi} \int_0^{\pi} \cos^2 u \cdot (1 - \cos^2 v) \cdot \sin v \cdot dv\, du$$

Change of variable

$w = \cos v$

$\frac{dw}{dv} = -\sin v$

$dv = \frac{-dw}{\sin v}$

Change of limits

$v = 0, \quad w = 1$

$v = \pi, \quad w = -1$

$$= \int_0^{2\pi} \int_1^{-1} \cos^2 u \cdot (1 - w^2) \cdot -dw\, du$$

$$= \int_0^{2\pi} \left[\cos^2 u \cdot \left(w - \tfrac{1}{3} w^3 \right) \right]_{-1}^{1} du$$

$$= \int_0^{2\pi} \frac{4}{3} \cos^2 u \, du$$

$$= \int_0^{2\pi} \frac{4}{3} \cdot \frac{1}{2} \left[1 + \cos(2u) \right] du$$

$$= \frac{2}{3} \left[u + \frac{1}{2} \sin(2u) \right]_0^{2\pi}$$

$$= \frac{4\pi}{3}.$$

② Evaluate $\iint_S y \, dS$ where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

Take $\bar{S}(u,v) = (u, v, u+v^2)$

for $0 \leq u \leq 1$, $0 \leq v \leq 2$.

Then $\dfrac{\partial \bar{S}}{\partial u} = (1, 0, 1)$

$\dfrac{\partial \bar{S}}{\partial v} = (0, 1, 2v)$

$$\frac{\partial \bar{S}}{\partial u} \times \frac{\partial \bar{S}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2v \end{vmatrix}$$

$$= \hat{i}(-1) + \hat{j}(-2v) + \hat{k}(1)$$

$$= (-1, -2v, 1)$$

$$\left| \frac{\partial \bar{S}}{\partial u} \times \frac{\partial \bar{S}}{\partial v} \right| = \sqrt{1+1+4v^2}$$

$f(\bar{S}) = v$

So $$\iint_S y \, dS = \int_0^1 \int_0^2 v\sqrt{2+4v^2} \, dv \, du$$

$$= \int_0^1 \left[\frac{1}{12} (2+4v^2)^{3/2} \right]_0^2 du$$

$$= \int_0^1 \frac{18^{3/2}}{12} - \frac{2^{3/2}}{12} \, du = \int_0^1 \frac{13\sqrt{2}}{3} \, du = \frac{13\sqrt{2}}{3}$$