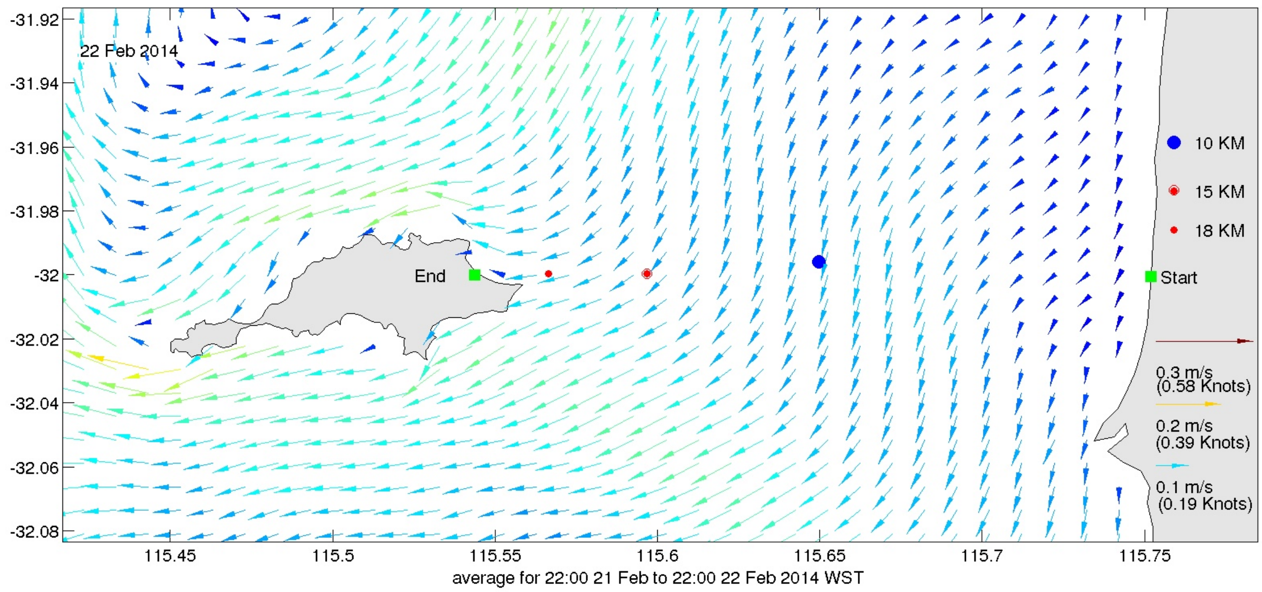
Path integrals of vector fields

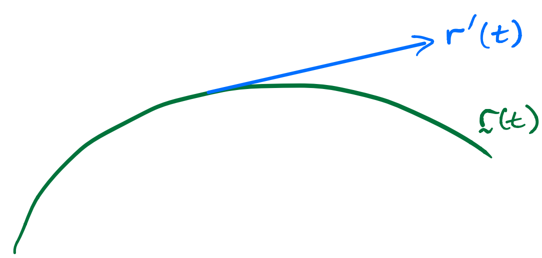
Ocean surface current forecast for the Rottnest channel swim



How much help do you get from the current along each path?

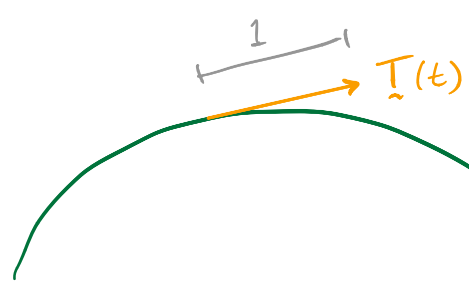
How much sideways drift do you need to correct for to stay on course?

Path: $\underset{\sim}{r}(t) = (x(t), y(t))$ has velocity $\underset{\sim}{r}'(t) =$



The unit tangent vector along $\underset{\sim}{r}(t)$ is $\underset{\sim}{T}(t) =$

$(\underset{\sim}{r}'(t) \neq 0)$



since $\underset{\sim}{T}(t)$ is a unit vector:

$$\|T(t)\| = 1$$
$$= 1$$
$$= 1$$

differentiating each side and using the product rule

$$= 0$$
$$= 0$$
$$= 0$$

so $\underset{\sim}{T}'(t)$ is perpendicular to $\underset{\sim}{r}'(t)$!

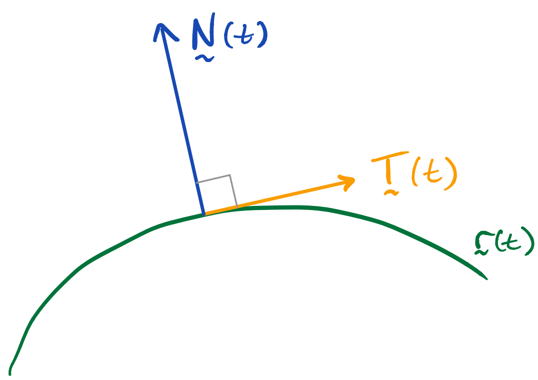
This means it is normal to the curve so we define the

normal vector

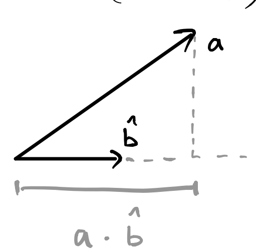
$\underset{\sim}{N}(t) =$

unit normal vector

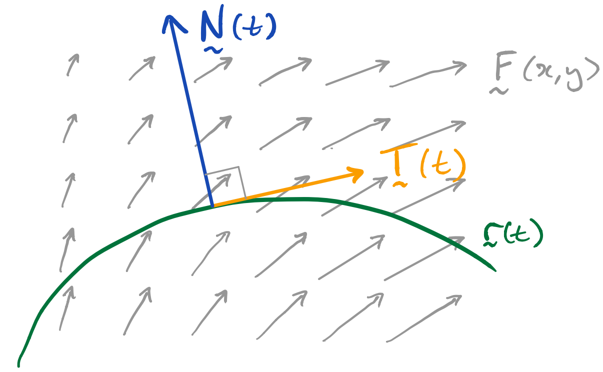
$\underset{\sim}{n}(t) =$



Recall: the dot product $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$, where $\hat{\underset{\sim}{b}}$ is a unit vector, gives us the component (amount) of $\underset{\sim}{a}$ in the direction $\hat{\underset{\sim}{b}}$



So if $\underset{\sim}{r}(t) = (x(t), y(t))$ represents the path of an object in the presence of a vector field (eg: ocean current) $\underset{\sim}{F}(x,y)$



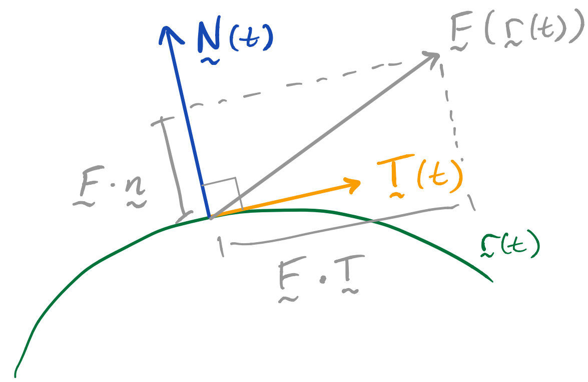
then the tangential component of F at time t along $\underset{\sim}{r}(t)$

'help' from the current.

and the normal component of F is

$$\underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \underset{\sim}{n}(t)$$

sideways drift



Integrating

Both components are scalar quantities which can be integrated (summed up) along the curve $C = \{ \underset{\sim}{r}(t) \in \mathbb{R}^2 \}$

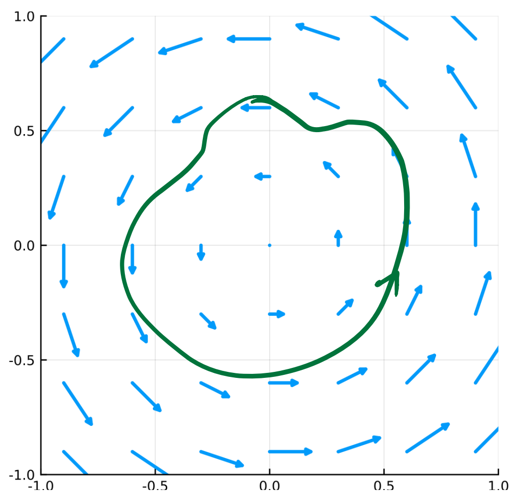
tangential

$$\int \underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \underset{\sim}{T}(t) \, \| \underset{\sim}{r}'(t) \| \, dt = \int \underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \underset{\sim}{r}'(t) \, dt$$

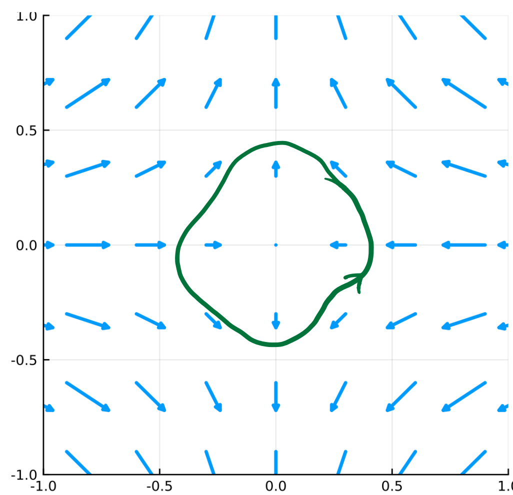
this quantity is called the circulation of $\underset{\sim}{F}$ along C

the name only really makes sense when C is closed, in which case this integral tells us how much the vector field circulates around C

positive circulation



no circulation



When $\underset{\sim}{F}$ represents a force this integral is the work done by the force acting on an object which traverses C.

$$\sum_i \underbrace{\underset{\sim}{F}(\underset{\sim}{r}(t_i)) \cdot \underset{\sim}{T}(t_i)}_{\text{component of } \underset{\sim}{F} \text{ which does work}} \underbrace{\| \underset{\sim}{r}'(t_i) \| \Delta t}_{\text{small displacement}} \xrightarrow{\lim} \int \underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \underset{\sim}{T}(t) \, \| \underset{\sim}{r}'(t) \| \, dt$$

$$= \text{Work}$$

Normal

$$\int \underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \underset{\sim}{n}(t) \, \|r'(t)\| \, dt = \int \underset{\sim}{F}(\underset{\sim}{r}(t)) \cdot \underline{N}(t) \, dt$$

this simplification is not obvious, requires calculating $\|T'(t)\|$.

this quantity is called the ~~total~~ flux across the curve

- imagine C represents a shallow fishing net, how much water is passing through per second?