## Differential equations

In a vast number of situations a mathematical model of a system or process will result in an equation (or set of equations) involving not only functions of the dependent variables but also derivatives of some or all of those functions with respect to one or more of the variables. Such equations are called *differential equations*.

The simplest situation is that of a single function of a single independent variable, in which case the equation is referred to as an *ordinary differential equation*.

A situation in which there is more than one independent variable will involve a function of those variables and an equation involving partial derivatives of that function is called a *partial differential equation*.



Notationally, it is easy to tell the difference. For example, the equation

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = f^2,$$

is a partial differential equation to be solved for $f(x, y)$, whereas

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} + 2f = x^4,$$

is an ordinary differential equation to be solved for $f(x)$.

The *order* of a differential equation is the degree of the highest derivative that occurs in it. The partial differential equation above is first-order, while the ordinary differential equation is second-order.

An important class of differential equations are those referred to as *linear*, where neither the function nor its derivatives occur in products, powers or nonlinear functions.



## Solutions of differential equations

When asked to solve an algebraic equation, for example $x^2 - 3x + 2 = 0$, we expect the answers to be numbers.

The situation with differential equations is much more difficult because we are being asked to find functions that will satisfy the given equation.

Unlike algebraic equations, which only have a discrete set of solutions (for example, $x^2 - 3x + 2 = 0$ only has the solutions $x = 1$ or 2), differential equations can have whole families of solutions.

For example, $y = Ce^{3x}$ satisfies the ordinary differential equation

$$\frac{dy}{dx} = 3y,$$

for *any* value of the constant $C$.



## Verification of solutions of differential equations

To get a feel for things (and to practice our algebra) we will have a quick look at the relatively simple procedure of verifying solutions of differential equations by way of a few examples.

*Example*

To verify that

$$y(x) = C_1 e^{2x} + C_2 e^{-2x} - 2\cos x - 5x \sin x,$$

is a solution of the ordinary differential equation

$$\frac{d^2 y}{dx^2} - 4y = 25x \sin x,$$

for any value of the constants $C_1$ and $C_2$, we need to calculate $\dfrac{d^2 y}{dx^2}$.

In order to do this we need the product rule to differentiate $x\sin x$. It gives

$$\frac{d}{dx}(x\sin x) = \sin x + x\cos x,$$

and

$$\frac{d^2}{dx^2}(x\sin x) = \frac{d}{dx}(\sin x + x\cos x) = 2\cos x - x\sin x.$$

Hence

$$\frac{d^2y}{dx^2} = 4C_1e^{2x} + 4C_2e^{-2x} - 8\cos x + 5x\sin x.$$

Then

$$\frac{d^2y}{dx^2} - 4y = \left[4C_1e^{2x} + 4C_2e^{-2x} - 8\cos x + 5x\sin x\right] - 4\left[C_1e^{2x} + C_2e^{-2x} - 2\cos x - 5x\sin x\right] = 25x\sin x,$$

as required.



*Example*

To verify that both

$$f(x,y) = xy - \frac{1}{2}y^2 \quad \text{and} \quad f(x,y) = \sin(y-x) + \frac{1}{2}x^2,$$

are solutions of the partial differential equation

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = x,$$

in each case we need to calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

For $f(x,y) = xy - \frac{1}{2}y^2$ we have

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x - y \quad \Rightarrow \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = y + x - y = x.$$



For $f(x,y) = \sin(y-x) + \frac{1}{2}x^2$ we have

$$\frac{\partial f}{\partial x} = -\cos(y-x) + x \quad \text{and} \quad \frac{\partial f}{\partial y} = \cos(y-x)$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = -\cos(y-x) + x + \cos(y-x) = x.$$

In both cases we have verified the solution of the partial differential equation.

In this Unit we will look as methods to solve *first-order separable* and *first-order linear* ordinary differential equations, and *second-order linear constant coefficient* ordinary differential equations.

The theory of partial differential equations is outside the scope of this Unit.



## First-order ordinary differential equations

Most first-order ordinary differential equations can be expressed (by algebraic re-arrangement if necessary) in the form

$$\frac{dy}{dx} = f(x,y),$$

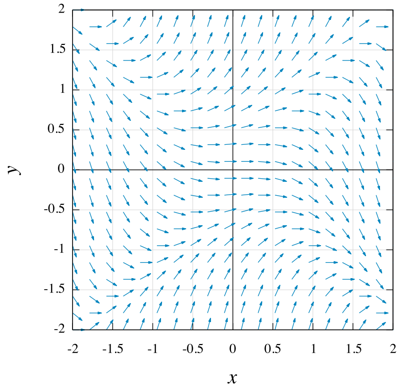
where the function $f(x,y)$ is known, and we are asked to find the solution $y(x)$.

## Direction fields

The differential equation $\dfrac{dy}{dx} = f(x,y)$ means that for any point in the $xy-$plane (where $f$ is defined) we can evaluate the gradient $\dfrac{dy}{dx}$ and represent this graphically by means of a small arrow representing the vector $\left(1, \dfrac{dy}{dx}\right)$.

If we do this for a whole grid of points in the $xy$-plane and place all of the arrows on the same plot we produce what is called a *direction field* or a *slope field*.

The figure below displays the direction field in the case where $f(x,y) = y^2 - x^2$.






A solution of $\dfrac{dy}{dx} = f(x,y)$ is a function relating $y$ and $x$ which geometrically is a curve in the $xy-$plane.

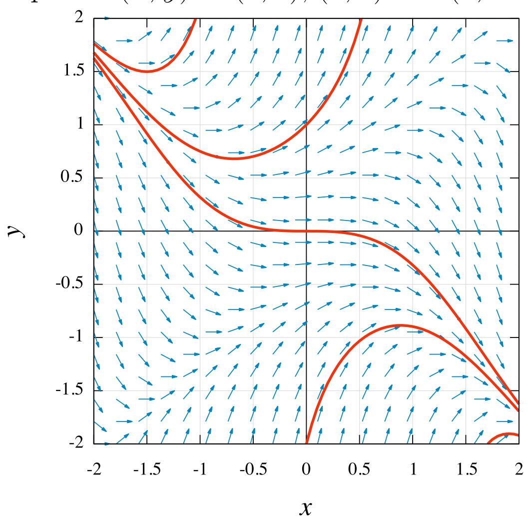
Since this solution satisfies the differential equation, the curve is such that its gradient is the same as the direction field vector at any point on the curve.

That is, the direction field is a collection of arrows that are tangential to the solution curves.

This observation enables us to roughly sketch solution curves without actually solving the differential equation, as long as we have a device to plot the direction field. We can indeed sketch many such curves (called a *family of solution curves*) superimposed on the same direction field.



The direction field of $\dfrac{dy}{dx} = y^2 - x^2$ along with three (disjoint) solution curves through the points $(x,y) = (0,1)$, $(0,0)$ and $(0,-2)$ is shown below.






Note that we will *not* be able to solve the differential equation $\dfrac{dy}{dx} = y^2 - x^2$ using the techniques we will cover in this unit – this differential equation is known as a *Ricatti differential equation*, which are notoriously difficult to solve!

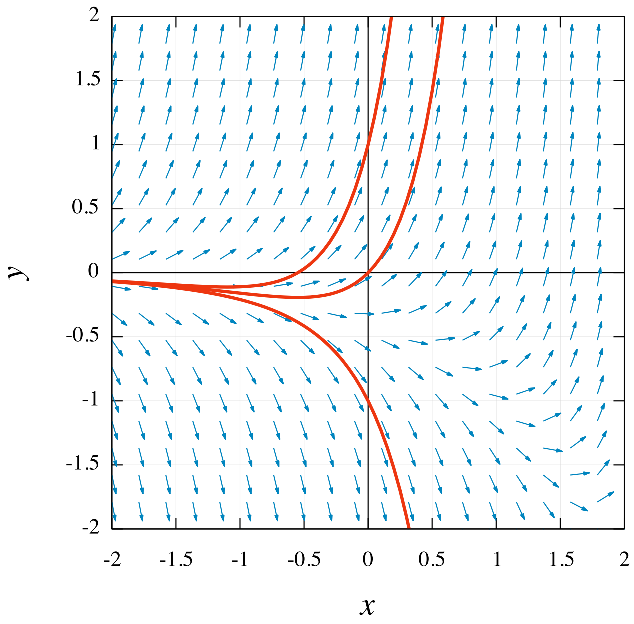
*Example*

The direction field of the differential equation

$$\frac{dy}{dx} = 3y + e^x,$$

along with three solution curves are shown in the figure below.

The top curve is the solution that goes through $(x,y) = (0,1)$, the middle curve is the solution that goes through $(x,y) = (0,0)$ and the bottom curve is the solution that goes through $(x,y) = (0,-1)$.


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
x

THE UNIVERSITY OF WESTERN AUSTRALIA

SCHOOL OF MATHEMATICS AND STATISTICS

# MATH1011 MULTIVARIABLE CALCULUS LECTURE NOTES WEEK 12

## First-order ordinary differential equations

A general first-order ordinary differential equation ($1^{\text{st}}$-order ODE, or just $1^{\text{st}}$-order DE) may be written in the form

$$\frac{dy}{dx} = f(x, y),$$

where $f(x, y)$ is an arbitrary (but known!) functions of $x$ and $y$.

We wish to find a solution $y(x)$ that satisfies the DE.



## Separable $1^{\text{st}}$-order DE's

A $1^{\text{st}}$-order DE is called *separable* provided that the function $f(x, y)$ may be written as the **product** of a function of $x$ and a function of $y$, that is $f(x, y) = F(x)G(y)$.

Thus the variables $x$ and $y$ can be "separated" and placed on opposite sides of the equation; that is, given

$$\frac{dy}{dx} = F(x)G(y).$$

Then by thinking of the derivative $\frac{dy}{dx}$ as a fraction we have

$$\frac{1}{G(y)}\,dy = F(x)\,dx,$$



and then each side can be integrated, so that

$$\int \frac{1}{G(y)}\,dy = \int F(x)\,dx + C,$$

where the arbitrary integration constant $C$ includes the constants from both integrals.

We then solve this equation (if possible) for $y$, which yields the *general solution* of the differential equation.

If we can **uniquely** solve for $y$, then the solution is called the *explicit solution* of the differential equation.

If we cannot uniquely solve for $y$, then the solution is called the *implicit solution* of the differential equation.



*Example*

Consider the DE

$$\frac{dy}{dx} = \sqrt{xy}.$$

Notice that

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{xy}\\ &= \sqrt{x}\sqrt{y}\\ &= x^{\frac{1}{2}}y^{\frac{1}{2}},\end{aligned}$$

so the DE is separable – separating $x$ and $y$ we have

$$y^{-\frac{1}{2}}\,dy = x^{\frac{1}{2}}\,dx,$$

and integrating both sides we have

$$2y^{\frac{1}{2}} = \tfrac{2}{3}x^{\frac{3}{2}} + C,$$

which is the implicit solution.

This can be uniquely solved for $y$, so the explicit solution is

$$y = \left(\tfrac{1}{3}x^{\frac{3}{2}} + C\right)^2,$$

where we have arbitrarily re-named the integration constant from $\dfrac{1}{2}C$ to just $C$.

**Example set 1 – week 12**



## $1^{\text{st}}$-order linear DE's

A $1^{\text{st}}$-order *linear* DE in one that may be written in the following *standard form*

$$\frac{dy}{dx} + f(x)y = g(x),$$

where $f(x)$ and $g(x)$ are arbitrary functions of $x$ only. Note that if $g(x) \neq 0$, the DE is <u>not</u> separable.

To solve such a DE, we multiply both sides by a function $I(x)$ such that the L.H.S. may be written

$$I\left(\frac{dy}{dx} + fy\right) = \frac{d}{dx}(Iy),$$

thus allowing the L.H.S. to be integrated – hence the function $I(x)$ is called an *integrating factor*.



**<u>If</u>** an integrating factor $I(x)$ can be found, then the general solution is

$$\frac{d}{dx}(Iy) = Ig \Rightarrow Iy = \int Ig\,dx + C \Rightarrow y(x) = \frac{1}{I(x)}\int I(x)g(x)\,dx + \frac{C}{I(x)}.$$

How to we find the function $I(x)$? Since

$$I\left(\frac{dy}{dx} + fy\right) = \frac{d}{dx}(Iy),$$

we have by expanding the L.H.S. and using the product rule on the R.H.S, that

$$I\frac{dy}{dx} + Ify = y\frac{dI}{dx} + I\frac{dy}{dx} \Rightarrow Ify = y\frac{dI}{dx} \Rightarrow \frac{dI}{dx} = If.$$

This is a separable DE for $I(x)$, with solution



$$\frac{1}{I}\,dI = f\,dx \Rightarrow \ln(I) = \int f\,dx + C \Rightarrow I = \exp\left(\int f\,dx + C\right).$$

We want the simplest possible solution for $I(x)$, so we set $C = 0$.

Hence the integrating factor $I(x) = \exp\left(\displaystyle\int f(x)\,dx\right)$.

Note that $\exp x$ is just another way of writing $e^x$ but has the advantage that the "power" $x$ is not a small superscript.

When learning the method it is instructive to follow through each step in the process in order to gain a better understanding of how it works.

Having said that, the general solution strategy is as follows:

1. Write the linear first-order differential equation in standard form $\dfrac{dy}{dx} + f(x)y = g(x)$ and identify the functions $f(x)$ and $g(x)$.
2. Find the integrating factor $I(x) = \exp\left(\int f(x)\,dx\right)$, omitting the integration constant.
3. Find $\int I(x)g(x)\,dx$, omitting the integration constant.
4. The general solution is then $y(x) = \dfrac{1}{I(x)}\int I(x)g(x)\,dx + \dfrac{C}{I(x)}$.



*Example*

Consider the DE

$$x\frac{dy}{dx} - y = x^2e^x.$$

In standard form we have

$$\frac{dy}{dx} - \frac{1}{x}y = xe^x,$$

so $f(x) = -\dfrac{1}{x}$ and $g(x) = xe^x$. Then

$$\int f(x)\,dx = \int -\frac{1}{x}\,dx = -\ln x = \ln\left(x^{-1}\right),$$

and hence

$$I(x) = \exp\left(\int f(x)\,dx\right) = e^{\ln(x^{-1})} = x^{-1}.$$



Then

$$\int I(x)g(x)\,dx = \int \left(x^{-1}\right)(xe^x)\,dx = \int e^x\,dx = e^x,$$

and the general solution is therefore

$$\begin{aligned} y(x) &= \frac{1}{I(x)}\int I(x)g(x)\,dx + \frac{C}{I(x)} \\ &= \left(\frac{1}{x^{-1}}\right)(e^x) + \frac{C}{x^{-1}} \\ &= xe^x + Cx. \end{aligned}$$

## Example set 2 – week 12



## Initial conditions

The values of any constants of integration that arise when we solve differential equations can be determined by making use of other conditions (or restrictions) placed on the problem.

For first-order differential equations, these conditions are called *initial conditions* and the combined differential equation plus initial condition is called an *initial value problem*.

The solution of an initial value problem is sometimes called a *particular solution* of the ODE, as it will not have any arbitrary constants, unlike the *general solution* of the ODE which actually corresponds to an infinite family of solutions due to the presence of arbitrary constants.

*Example*

To solve the initial value problem

$$\frac{dy}{dx} = \frac{x^2}{y} \quad , \quad y(1) = 4,$$

We observe that the differential equation is separable. The solution is:

$$\int y\, dy = \int x^2\, dx \quad \Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}x^3 + C,$$

which implies

$$y(x) = \pm\sqrt{\frac{2}{3}x^3 + C},$$

where we have arbitrarily re-named the integration constant.

Notice that we have two different solutions to the differential equation, one positive and one negative.



The initial condition $y(1) = 4$ allows us to eliminate the negative solution, so we are left with

$$y(x) = \sqrt{\frac{2}{3}x^3 + C},$$

and substituting into this $y = 4$ and $x = 1$ gives

$$4 = \sqrt{\frac{2}{3} + C} \quad \Rightarrow \quad C = \frac{46}{3} \quad \Rightarrow \quad y(x) = \sqrt{\frac{2x^3 + 46}{3}}.$$

**Example set 3 – week 12**



## Second-order ordinary differential equations

A general second-order differential equation ($2^{\text{nd}}$-order DE) may be written in the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right),$$

where $f(x, y, y')$ is an arbitrary (but known!) function of $x$, $y$ and $y'$, and we wish to find a solution $y(x)$ that satisfies the given DE.

A $2^{\text{nd}}$-order *linear* differential equation is one that may be written

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x),$$

where $p(x)$, $q(x)$ and $g(x)$ are arbitrary functions of $x$.



A $2^{\text{nd}}$-order linear DE is said to be *homogeneous* if $g(x) = 0$.

Otherwise the DE is *nonhomogeneous* and $g(x)$ is called the *nonhomogeneous term*.

If $p(x) = p$, $q(x) = q$ are constant functions and $g(x) = 0$, then we have a $2^{\text{nd}}$-order linear homogeneous DE with *constant coefficients*

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

otherwise the DE is said to have *variable coefficients*.

Since two integrations are required to find a solution of a $2^{\text{nd}}$-order DE and each integration produces an arbitrary integration constant, the general solution $y(x)$ will contain two integration constants $C_1$ and $C_2$.

The *principal of superposition* states that if $y_1$ and $y_2$ are both solutions of a $2^{\text{nd}}$-order linear homogeneous DE, then so is the function

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

See the Unit Reader for a proof of this.

For the solution $y = C_1 y_1 + C_2 y_2$ to be the general solution of the DE, the solutions $y_1$ and $y_2$ must also be *linearly independent*.

Two functions $y_1$ and $y_2$ are called linearly independent if they are not constant multiples of one another, while they are *linearly dependent* if there exists constants $C_1$ and $C_2$, not both zero, such that

$$C_1 y_1(x) + C_2 y_2(x) = 0,$$

for all $x$.



A simple way to check if two solutions $y_1$ and $y_2$ are linearly independent is to calculate a function called the *Wronskian* of $y_1$ and $y_2$, denoted by $W[y_1, y_2](x)$, which is defined by

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx},$$

where as usual $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is the determinant.

Then $y_1$ and $y_2$ are linearly dependent if $W[y_1, y_2](x) = 0$ for all $x$, while they are linearly independent if $W[y_1, y_2](x) \neq 0$.

To prove this, we need some concepts of *linear algebra* which are covered in MATH1012, hence it is omitted here.



The conclusion from all this is: given a second-order linear homogeneous differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

for arbitrary constants $C_1$ and $C_2$, where both $y_1$ and $y_2$ are solutions of the differential equation.

The general solution includes **every possible solution** of the differential equation provided the functions $y_1$ and $y_2$ are linearly independent, that is the Wronskian

$$W[y_1, y_2](x) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} \neq 0.$$

**Example set 4 – week 12**

THE UNIVERSITY OF WESTERN AUSTRALIA

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# MATH1011 MULTIVARIABLE CALCULUS
# LECTURE NOTES WEEK 13

## Second-order linear homogeneous differential equations with constant coefficients

The general form of a $2^{\text{nd}}$-order linear homogeneous DE with constant coefficients is

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

where $p$ and $q$ are constants. We attempt to find the general solution by assuming a solution of the form $y = e^{mx}$, where $m$ is a constant to be determined.



Assuming a solution of the form $y = e^{mx}$ yields

$$e^{mx}(m^2 + pm + q) = 0 \Rightarrow m^2 + pm + q = 0.$$

This is called the *characteristic equation* of the DE.

Since it is a quadratic in $m$, it has two roots

$$m_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \quad , \quad m_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Hence there are three cases to consider, depending on whether the discriminant $p^2 - 4q$ is positive, negative or zero.



### *Case 1. Two real roots*

In this case the discriminant is positive and we have two real distinct roots $m_1$ and $m_2$.

Then $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are two solutions of the DE, and the Wronskian is

$$W[y_1, y_2](x) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = (m_2 - m_1)e^{(m_1 + m_2)x},$$

which is never zero since $m_1 \neq m_2$.

Then the general solution of the DE is

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$



### *Case 2. Complex conjugate roots*

In this case the discriminant is negative and we have two complex roots $m_1 = a + ib$ and $m_2 = a - ib$ that are complex conjugates of each other, where $a = -\frac{1}{2}p$ and $b = \frac{1}{2}\sqrt{4q - p^2}$. Then $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are two solutions of the DE, and the Wronskian is again never zero since $m_1 \neq m_2$. The general solution of the DE is then

$$y(x) = C_1 e^{(a+ib)x} + C_2 e^{(a-ib)x}.$$

Recalling *Euler's formula* $e^{ix} = \cos x + i \sin x$, we have

$$\begin{aligned} y(x) &= C_1 e^{ax} e^{ibx} + C_2 e^{ax} e^{-ibx} \\ &= C_1 e^{ax} [\cos(bx) + i \sin(bx)] + C_2 e^{ax} [\cos(bx) - i \sin(bx)] \\ &= (C_1 + C_2) e^{ax} \cos(bx) + i(C_1 - C_2) e^{ax} \sin(bx) \\ &= C_1 e^{ax} \cos(bx) + C_2 e^{ax} \sin(bx), \end{aligned}$$

where we have arbitrarily re-named the two integration constants.

## *Case 3. Equal roots*

In this case the discriminant is zero and we have one repeated root $m = -\frac{1}{2}p$, so we only know "half" of the general solution, namely $y_1 = e^{mx}$.

How to we find the other "half" of the solution, namely $y_2$?

If we let $y(x) = v(x)y_1(x) = v(x)e^{-\frac{1}{2}px}$ for some function $v(x)$ to be found, then using the product rule we have

$$\frac{dy}{dx} = \left(e^{-\frac{1}{2}px}\right)\frac{dv}{dx} + v\left(-\frac{1}{2}pe^{-\frac{1}{2}px}\right) = e^{-\frac{1}{2}px}\left(\frac{dv}{dx} - \frac{1}{2}pv\right),$$

and using the product rule again we find that

$$\frac{d^2y}{dx^2} = e^{-\frac{1}{2}px}\left(\frac{d^2v}{dx^2} - p\frac{dv}{dx} + \frac{1}{4}p^2v\right).$$



Then the differential equation becomes

$$e^{-\frac{1}{2}px}\left(\frac{d^2v}{dx^2} - p\frac{dv}{dx} + \frac{1}{4}p^2v\right) + pe^{-\frac{1}{2}px}\left(\frac{dv}{dx} - \frac{1}{2}pv\right) + qve^{-\frac{1}{2}px} = 0,$$

which simplifies to

$$\frac{d^2v}{dx^2} + \left(-\frac{1}{4}p^2 + q\right)v = 0.$$

Since $p^2 - 4q = 0$, the coefficient of $v$ in the above equation is zero, so we have

$$\frac{d^2v}{dx^2} = 0$$

which can be integrated twice to give

$$v(x) = C_1 + C_2x.$$

Therefore

$$y(x) = v(x)y_1(x) = (C_1 + C_2x)e^{-\frac{1}{2}px} = C_1e^{-\frac{1}{2}px} + C_2xe^{-\frac{1}{2}px}.$$



Then the second linearly independent solution of the differential equation is therefore $y_2 = xe^{-\frac{1}{2}px}$.

**Note:** This is an example of a process called *reduction of order*, which is a way to "build" the general solution of a second-order linear differential equation provided we can find a single solution $y_1$.

So, if the characteristic equation has only one root $m$, and the general solution of the differential equation is

$$y(x) = C_1e^{mx} + C_2xe^{mx}.$$

Notice that the Wronskian is

$$W[y_1, y_2](x) = \begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & (1+mx)e^{mx} \end{vmatrix} = e^{2mx},$$

which is never zero since $m \neq 0$.



*Summary*

In summary, to find the general solution of a linear homogeneous second-order ordinary homogeneous differential equation with constant coefficients

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

where $p$ and $q$ are constants, find the roots of the characteristic equation

$$m^2 + pm + q = 0.$$

1. If the roots $m_1$ and $m_2$ are real and unequal, then the general solution is $y(x) = C_1e^{m_1x} + C_2e^{m_2x}$.
2. If the roots are complex conjugates $a \pm ib$, then the general solution is $y(x) = C_1e^{ax}\cos(bx) + C_2e^{ax}\sin(bx)$.
3. If there is a single (or repeated) root $m$, then the general solution is $y(x) = C_1e^{mx} + C_2xe^{mx}$.

**Example set 1 – week 13**

## Linear nonhomogeneous second-order ordinary differential equations with constant coefficients

Consider a linear nonhomogeneous second-order ordinary differential equations with constant coefficients

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = g(x),$$

where $p$ and $q$ are constants and the nonhomogeneous term $g(x)$ is an arbitrary function of $x$.

For this differential equation we also consider the corresponding homogeneous differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

with general solution $y_c$, which we call the *complementary solution*.



*Definition – particular solution*

A *particular solution* $y_p$ of the nonhomogeneous differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = g(x),$$

is a specific function that contains no arbitrary constants and satisfies the differential equation.

*Theorem – general solution of a nonhomogeneous differential equation*

The general solution of a linear nonhomogeneous second-order ordinary differential equations with constant coefficients

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = g(x),$$



is

$$y(x) = y_c(x) + y_p(x),$$

where $y_p$ is a particular solution of the nonhomogeneous differential equation and $y_c$ is the general solution of the corresponding homogeneous differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0.$$

There are two methods to find $y_p(x)$, either the *method of undetermined coefficients* (a very specific method) or *variation of parameters* (a completely general method).



## Method of undetermined coefficients

The method of undetermined coefficients can be applied to a $2^{\text{nd}}$-order linear nonhomogeneous DE with constant coefficients when the nonhomogeneous term is:

- A polynomial;
- A linear combination of sines and cosines;
- An exponential function; or
- A combination of sums, differences and products of the above functions.

The idea behind this method is that the derivative of a polynomial is a polynomial, that of a trig function is a trig function, and that of an exponential function is an exponential function, meaning that we can make an intelligent guess for the form of $y_p(x)$.

| Nonhomogeneous term $g(x)$ | Form of trial particular solution $y_p(x)$ |
|---|---|
| $a_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ | $A_n(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$ |
| $a_n(x)e^{\alpha x}$ | $A_n(x)e^{\alpha x}$ |
| $a_n(x)\sin(\beta x)$ or $a_n(x)\cos(\beta x)$ | $A_n(x)\sin(\beta x) + B_n(x)\cos(\beta x)$ |
| $a_n(x)e^{\alpha x}\sin(\beta x)$ or $a_n(x)e^{\alpha x}\cos(\beta x)$ | $e^{\alpha x}\left[A_n(x)\sin(\beta x) + B_n(x)\cos(\beta x)\right]$ |

We formulate a guess for $y_p$ using the above table and the following rules:

- **Basic rule:** If $g(x)$ is one of the functions listed in the first column, substitute the corresponding function from the second column and determine the unknown constants by equating coefficients.



- **Modification rule:** If a term in the choice for $y_p$ is a solution of the homogeneous equation, then multiply this term by $x$.

  *This should come as no surprise – remember that the second solution of a second-order homogeneous differential equation with constant coefficients for the case of an equal root of the characteristic equation was just $x$ times the first solution.*

- **Sum rule:** If $g(x)$ is a sum of functions listed in the first column, then substitute the corresponding sum of functions from the second column and solve for the unknown coefficients by equating coefficients.

**Example set 2 – week 13**



## Variation of parameters

In cases where when the method of undetermined coefficients cannot be applied, either because the nonhomogeneous term is not of the right type or the coefficients of the DE are not constant, a more general method called variation of parameters may be used to find $y_p$.

Consider the complimentary solution $y_c = C_1 y_1 + C_2 y_2$ of the general $2^{\text{nd}}$-order linear homogeneous DE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

To find a particular solution $y_p$ of the corresponding nonhomogeneous DE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x),$$



we replace the integration constants $C_1$ and $C_2$ in the complementary solution with unknown functions $u_1(x)$ and $u_2(x)$ and suppose that this is $y_p$; that is, we set

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

From the product rule we have

$$\frac{dy_p}{dx} = y_1\frac{du_1}{dx} + u_1\frac{dy_1}{dx} + y_2\frac{du_2}{dx} + u_2\frac{dy_2}{dx} = u_1\frac{dy_1}{dx} + u_2\frac{dy_2}{dx} + \left(y_1\frac{du_1}{dx} + y_2\frac{du_2}{dx}\right).$$

This is a nasty-looking expression, so lets set the term in brackets equal to zero, that is

$$y_1\frac{du_1}{dx} + y_2\frac{du_2}{dx} = 0,$$

then the first derivative of $y_p$ is the not-so-nasty-looking

$$\frac{dy_p}{dx} = u_1\frac{dy_1}{dx} + u_2\frac{dy_2}{dx}.$$

Differentiating again using the product rule we have

$$\frac{d^2y_p}{dx^2} = \frac{dy_1}{dx}\frac{du_1}{dx} + u_1\frac{d^2y_1}{dx^2} + \frac{dy_2}{dx}\frac{du_2}{dx} + u_2\frac{d^2y_2}{dx^2},$$

and hence the nonhomogeneous DE becomes

$$\frac{dy_1}{dx}\frac{du_1}{dx} + u_1\frac{d^2y_1}{dx^2} + \frac{dy_2}{dx}\frac{du_2}{dx} + u_2\frac{d^2y_2}{dx^2} + p\left(u_1\frac{dy_1}{dx} + u_2\frac{dy_2}{dx}\right) + q\left(u_1y_1 + u_2y_2\right) = g,$$

which may be written

$$u_1\left(\frac{d^2y_1}{dx^2} + p\frac{dy_1}{dx} + qy_1\right)^{\nearrow 0} + u_2\left(\frac{d^2y_2}{dx^2} + p\frac{dy_2}{dx} + qy_2\right)^{\nearrow 0} + \frac{dy_1}{dx}\frac{du_1}{dx} + \frac{dy_2}{dx}\frac{du_2}{dx} = g.$$



The terms in brackets are zero (Why?), so that we have two equations for the derivatives of the unknown functions $u_1$ and $u_2$, namely

$$y_1\frac{du_1}{dx} + y_2\frac{du_2}{dx} = 0 \quad \text{and} \quad \frac{dy_1}{dx}\frac{du_1}{dx} + \frac{dy_2}{dx}\frac{du_2}{dx} = g.$$

Solving these equations for $u_1'$ and $u_2'$ we find

$$\frac{du_1}{dx} = -\frac{y_2 g}{W[y_1, y_2]} \quad \text{and} \quad \frac{du_2}{dx} = \frac{y_1 g}{W[y_1, y_2]},$$

where $W[y_1, y_2]$ is the Wronskian of the homogeneous solutions $y_1$ and $y_2$.

By integrating these equations (omitting the integration constants) we can obtain the particular solution $y_p = u_1y_1 + u_2y_2$, and therefore the general solution $y = y_c + y_p$.

*Although we said this is a general method, there is no guarantee that these two equations* can *be integrated to find $u_1(x)$ and/or $u_2(x)$!*



If one of the terms in $y_p$ is already in $y_c$, we can "absorb" it into the integration constants in $y_c$.

Note that the examples that were solved in the previous section by the method of undetermined coefficients can also be solved by variation of parameters.

However, in the vast majority of cases if a nonhomogeneous differential equation can be solved by the method of undetermined coefficients, it will be much easier to use that method than to solve than the same problem using variation of parameters.

To see this, solve one of the examples we did previously using the method of undetermined coefficients using instead the method of variation of parameters – it will be a lot more work!



*Summary*

To find the general solution of a linear nonhomogeneous second-order ordinary differential equation with constant coefficients of general form

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = g(x),$$

where $p$ and $q$ are constants and $g(x)$ is an arbitrary function of $x$ by the method of variation of parameters:

1. Find the general solution $y_c(x) = C_1y_1(x) + C_2y_2(x)$ of the corresponding homogeneous differential equation
$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0.$$

2. Calculate the Wronskian

$$W[y_1, y_2](x) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}.$$

3. Let $\dfrac{du_1}{dx} = -\dfrac{y_2(x)g(x)}{W[y_1, y_2]}$ and $\dfrac{du_2}{dx} = \dfrac{y_1(x)g(x)}{W[y_1, y_2]}$.

4. Integrate these two equations to find $u_1(x)$ and $u_2(x)$, omitting the integration constants.

5. A particular solution of the nonhomogeneous differential is then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.

6. The general solution of the nonhomogeneous differential equation is then $y(x) = y_c(x) + y_p(x)$.

**Example set 3 – week 13**