MATH1011 Class Examples Week 12

Example set 1

① Solve the DE

$$\frac{dy}{dx} = y^2 \sin x$$

Here we have

$$\frac{dy}{y^2} = \sin x \, dx \Rightarrow y^{-2} dy = \sin x \, dx$$

and integrating both sides we have

$$-y^{-1} = -\cos x + C$$

which is the implicit solution.

Rearranging

$$y = \frac{1}{\cos x + C}$$

where we have arbitrarily renamed the integration constant.

This is the explicit solution.

② Solve the DE

$$\frac{dy}{dx} = \frac{y\cos x}{1+y^2}$$

Notice that $\dfrac{y\cos x}{1+y^2} = \left(\dfrac{y}{1+y^2}\right) \times (\cos x)$,

so we have

$$\left(\frac{1+y^2}{y}\right) dy = \cos x \, dx \implies \left(\frac{1}{y} + y\right) dy = \cos x \, dx$$

So

$$\ln y + \tfrac{1}{2} y^2 = \sin x + C$$

This is the implicit solution.

Note that we cannot find an explicit solution in this case, since it is not possible to solve

$$\ln y + \tfrac{1}{2} y^2$$

for y.

Example set 2

① Find the general solution of the DE

$$\frac{dy}{dx} = e^{-2x} - 3y.$$

In standard form we have

$$\frac{dy}{dx} + 3y = e^{-2x}$$

which is a linear DE where

$$f(x) = 3 \quad , \quad g(x) = e^{-2x}$$

Then

$$\int f\,dx = \int 3\,dx = 3x$$

$$I = \exp\left(\int f\,dx\right) = \exp(3x) = e^{3x}$$

$$\int Ig\,dx = \int e^{3x}.e^{-2x}\,dx$$

$$= \int e^{x}\,dx = e^{x}$$

Hence the general solution is

$$y = \frac{1}{I}\int Ig\,dx + \frac{C}{I} = \frac{1}{e^{3x}}.e^{x} + \frac{C}{e^{3x}}$$

$$= e^{-2x} + Ce^{-3x}$$

(2) Find the general solution of

$$x\frac{dy}{dx} - y = x\ln x$$

Standard form:

$$\frac{dy}{dx} - \frac{1}{x}.y = \ln x$$

so $f(x) = \frac{-1}{x}$, $g(x) = \ln x$

$$\int f\,dx = \int \frac{-1}{x}\,dx = -\ln x = \ln(x^{-1})$$

$$I = \exp\left(\int f\,dx\right) = \exp(\ln(x^{-1})) = x^{-1}$$

<u>NOT</u> $I = \exp(-\ln x) = -x$!!!

$$\int I g\,dx = \int \frac{1}{x}.\ln x\,dx$$

<u>Change of variable</u>

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$$= \int \frac{1}{x}.u.x\,du$$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 = \frac{1}{2}(\ln x)^2$$

So $y = \frac{1}{x^{-1}} \cdot \frac{1}{2}(\ln x)^2 + \frac{c}{x^{-1}} = \frac{1}{2}x(\ln x)^2 + Cx$

Example set 3

① Solve the initial value problem (IVP)

$$\frac{dy}{dx} = \sqrt{xy} \quad , \quad y(0) = 1$$

We have from the lecture notes the general solution

$$y = \left(\frac{1}{3}x^{3/2} + C\right)^2$$

When $x=0$, $y=1$: $1 = (0+C)^2$

$$\Rightarrow C^2 = 1$$

$$\Rightarrow C = \pm 1$$

Which value of C do we use?

Go back to the implicit solution:

$$2y^{1/2} = \frac{2}{3}x^{3/2} + C$$

When $x=0, y=1$: $2 = 0 + C \Rightarrow C = 2$

$$\Rightarrow 2y^{1/2} = \frac{2}{3}x^{3/2} + 2$$

$$\Rightarrow y = \left(\frac{1}{3}x^{3/2} + 1\right)^2 .$$

(2) Solve the IVP

$$\frac{dy}{dx} + \frac{2}{x} y = x^2, \quad y(1) = 0$$

Linear DE : $f(x) = \frac{2}{x}$, $g(x) = x^2$

$$\int f\,dx = \int \frac{2}{x}\,dx = 2\ln x = \ln(x^2)$$

$$I = \exp\left(\int f\,dx\right) = \exp(\ln(x^2)) = x^2$$

<u>NOT</u> $I = 2x$!!!

$$\int Ig\,dx = \int x^2 . x^2\,dx = \int x^4\,dx$$

$$= \frac{1}{5}x^5$$

$\therefore$ general solution is

$$y = \frac{1}{x^2} \cdot \frac{1}{5}x^5 + \frac{c}{x^2} = \frac{1}{5}x^3 + \frac{c}{x^2}$$

When $x = 1$, $y = 0$: $0 = \frac{1}{5} + c \Rightarrow c = -\frac{1}{5}$

$\therefore$ particular solution is

$$y = \frac{1}{5}x^3 - \frac{1}{5x^2}$$

Example set 4

① Determine if the following sets of functions are linearly dependent / independent

a) $y_1 = 2x$, $y_2 = 4x$

b) $y_1 = e^x$, $y_2 = e^{-x}$

c) $y_1 = x^{2/3}$, $y_2 = x$

a) $W[y_1, y_2](x) = \begin{vmatrix} 2x & 4x \\ 2 & 4 \end{vmatrix}$

$= (2x)(4) - (4x)(2)$

$= 8x - 8x = 0$

$\therefore$ they are linearly dependent

This was obvious since $y_2 = 2 \times y_1$.

b) $W[y_1, y_2](x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$

$= (e^x)(-e^{-x}) - (e^{-x})(e^x)$

$= -1 - 1 = -2 \neq 0$

$\therefore$ linearly independent.

c) $W[y_1, y_2](x) = \begin{vmatrix} x^{2/3} & x \\ 2/3\, x^{-1/3} & 1 \end{vmatrix}$

$$= x^{2/3} - \frac{2}{3}x^{2/3}$$

$$= \frac{1}{3}x^{2/3} \neq 0 \text{ unless } x = 0.$$

Since $W[y_1, y_2](x) \neq 0$ for <u>all</u> x, the functions are linearly independent.

(2) Show that $y_1 = x^{2/3}$ and $y_2 = x$ are solutions of the DE

$$3x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

For $y_1 = x^{2/3}$ the DE becomes

$$(3x^2)\left(\frac{-2}{9}x^{-4/3}\right) - (2x)\left(\frac{2}{3}x^{-1/3}\right) + (2)(x^{2/3})$$

$$= \frac{-2}{3}x^{2/3} - \frac{4}{3}x^{2/3} + 2x^{2/3} = 0 \text{ as required.}$$

For $y_2 = x$ the DE becomes

$$(3x^2)(0) - (2x)(1) + (2)(x) = -2x + 2x = 0$$

as required.

Since y_1 and y_2 are also linearly independent, the general solution of the DE is therefore

$$y(x) = C_1 x^{2/3} + C_2 x$$

for arbitrary constants C_1 and C_2

Note we saw that

$$W[y_1, y_2](x) = \frac{1}{3} x^{2/3}$$

which is obviously zero when $x=0$.

From the DE:

$$3x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$$

When $x=0$ we have

$$2y = 0 \Rightarrow y = 0$$

That is, when $x=0$ we only get the trivial solution $y=0$.

Special situations like this is something we should be mindful of.

MATH1011 Class Examples Week 13

Example set 1

① Find the general solution of the DE

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$$

Characteristic equation: $m^2 + 3m - 4 = 0$

$(m+4)(m-1) = 0$

$\therefore m = -4, 1$

$\therefore$ general solution is

$$y(x) = C_1 e^{-4x} + C_2 e^{x}$$

② Find the general solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Characteristic equation: $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$\therefore m = -1$ (repeated root)

$\therefore$ general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}.$$

(3) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 20y = 0$

Characteristic equation: $m^2 + 4m + 20 = 0$

$$m = \frac{-4 \pm \sqrt{16-80}}{2} = \frac{-4 \pm \sqrt{-64}}{2}$$

$$= \frac{-4 \pm 8i}{2} = -2 \pm 4i$$

$\therefore$ $m = a \pm ib$ where $a=-2$, $b=4$

$\therefore$ general solution is

$$y(x) = C_1 e^{-2x}\sin(4x) + C_2 e^{-2x}\cos(4x).$$

Example set 2

① Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2e^x$.

Homogeneous DE: $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Characteristic equation: $m^2 + 5y + 6 = 0$

$(m+3)(m+2) = 0$

$\therefore m = -3, -2$

$\therefore y_c(x) = C_1 e^{-3x} + C_2 e^{-2x}$

Since $g(x) = 2e^x$, try $y_p = Ae^x$

Check: y_p not in y_c ✓ OK

$\frac{dy_p}{dx} = Ae^x$, $\frac{d^2y_p}{dx^2} = Ae^x$

$\therefore$ nonhomogeneous DE becomes

$(Ae^x) + (5)(Ae^x) + (6)(Ae^x) = 2e^x$

$\Rightarrow 12Ae^x = 2e^x \Rightarrow A = \frac{1}{6}$

$\therefore$ general solution is

$y(x) = y_c + y_p = C_1 e^{-3x} + C_2 e^{-2x} + \frac{1}{6}e^x$.

② Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2x^2 + 3x$

From Q1: $y_c(x) = C_1 e^{-2x} + C_2 e^{-3x}$

$g(x) = 2x^2 + 3x \Rightarrow$ try $y_p = Ax^2 + Bx + C$

Check: y_p not in y_c ✓ ok

$\frac{dy_p}{dx} = 2Ax + B \quad , \quad \frac{d^2y_p}{dx^2} = 2A$

∴ nonhomogeneous DE becomes

$$(2A) + (5)(2Ax + B) + (6)(Ax^2 + Bx + C)$$
$$= 2x^2 + 3x$$

$$\Rightarrow 6Ax^2 + (10A + 6B)x + (2A + 5B + 6C)$$
$$= 2x^2 + 3x$$

Equating coefficients:

$6A = 2$
$10A + 6B = 3$
$2A + 5B + 6C = 0$

So $A = 1/3$, $B = -1/18$, $C = -7/108$

$$\therefore y(x) = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{3}x^2 - \frac{1}{18}x - \frac{7}{108}$$

3) Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 4\cos x$

$y_c(x) = C_1 e^{-2x} + C_2 e^{-3x}$ (from Q1).

$g(x) = 4\cos x \Rightarrow$ try $y_p = A\cos x + B\sin x$

<u>Check</u>: y_p not in y_c ✓ok

$$\frac{dy_p}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2y_p}{dx^2} = -A\cos x - B\sin x$$

$\therefore$ nonhomogeneous DE becomes

$$(-A\cos x - B\sin x) + 5(-A\sin x + B\cos x) + 6(A\cos x + B\sin x) = 4\cos x$$

$$\Rightarrow (5A + 5B)\cos x + (-5A + 5B)\sin x = 4\cos x$$

Equating coefficients:

$$5A + 5B = 4$$
$$-5A + 5B = 0$$

$$\Rightarrow A = \frac{2}{5}, \quad B = \frac{2}{5}$$

$\therefore\ y(x) = C_1e^{-2x} + C_2e^{-3x} + \frac{2}{5}\cos x + \frac{2}{5}\sin x$

(4) Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 3e^{-2x}$

$y_c(x) = C_1e^{-2x} + C_2e^{-3x}$ (from Q1)

$g(x) = 3e^{-2x} \Rightarrow$ try $y_p = Ae^{-2x}$

Check: y_p not in y_c X multiply by x

Try $y_p = Axe^{-2x}$

Check: y_p not in y_c ✓ ok

$\frac{dy_p}{dx} = Ae^{-2x} - 2Axe^{-2x}$

$\frac{d^2y_p}{dx^2} = -2Ae^{-2x} - 2A\left(e^{-2x} - 2xe^{-2x}\right)$

$= -4Ae^{-2x} + 4Axe^{-2x}$

$\therefore$ nonhomogeneous DE becomes

$\left(-4Ae^{-2x} + 4Axe^{-2x}\right) + 5\left(Ae^{-2x} - 2Axe^{-2x}\right)$

$+ 6\left(Axe^{-2x}\right) = 3e^{-2x}$

$\Rightarrow -4A + 4Ax + 5A - 10Ax + 6Ax = 3$

Hence $A = 3$, and

$$y(x) = C_1 e^{-2x} + C_2 e^{-3x} + 3xe^{-2x}$$

Note If we tried $y_p = Ae^{-2x}$ then

$$\frac{dy_p}{dx} = -2Ae^{-2x}, \quad \frac{d^2y_p}{dx^2} = 4Ae^{-2x}$$

and the nonhomogeneous DE would become

$$(4Ae^{-2x}) + 5(-2Ae^{-2x}) + 6Ae^{-2x} = 3e^{-2x}$$

$$4A - 10A + 6A = 3$$

$$0 = 3 \; !$$

Example set 3

① Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 3e^{-2x}$

This is the same as Q4 from Example Set 2.

We have $y_c(x) = C_1 e^{-2x} + C_2 e^{-3x}$

Take $y_1 = e^{-2x}$, $y_2 = e^{-3x}$

Then

$$W[y_1, y_2](x) = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}$$

$$= -3e^{-5x} + 2e^{-5x} = -e^{-5x}$$

Here we have $g(x) = 3e^{-2x}$

Then $\frac{du_1}{dx} = \frac{-y_2 g}{W} = \frac{-(e^{-3x})(3e^{-2x})}{(-e^{-5x})}$

$$= +3$$

So $u_1 = 3x$

$$\frac{du_2}{dx} = \frac{y_1 g}{W} = \frac{(e^{-2x})(3e^{-2x})}{(-e^{-5x})} = -3e^{x}$$

So $u_2 = -3e^{x}$

Then $y_p = u_1y_1 + u_2y_2 = (3x)(e^{-2x}) + (-3e^{x})(e^{-3x})$

$$= 3xe^{-2x} - 3e^{-2x}$$

Then $y(x) = y_c + y_p = C_1e^{-2x} + C_2e^{-3x} + 3xe^{-2x} - 3e^{-2x}$

$$= (C_1 - 3)e^{-2x} + C_2e^{-3x} + 3xe^{-2x}$$

$$= C_1e^{-2x} + C_2e^{-3x} + 3xe^{-2x}$$

Same as before!

(2) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{x}\ln x$

Homogeneous solution: $y_c(x) = C_1e^{x} + C_2xe^{x}$

Take $y_1 = e^{x}$, $y_2 = xe^{x}$

Then $W[y_1, y_2](x) = \begin{vmatrix} e^{x} & xe^{x} \\ e^{x} & xe^{x} + e^{x} \end{vmatrix}$

$$= (e^{x})(xe^{x} + e^{x}) - (xe^{x})(e^{x}) = e^{2x}$$

Here we have $g(x) = e^{x}\ln x$.

Then $\frac{du_1}{dx} = \frac{-y_2 g}{W} = -\frac{(xe^x)(e^x \ln x)}{(e^{2x})}$

$= -x \ln x$

Then using integration by parts we have

$u_1 = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2$ [Check this!]

Also, $\frac{du_2}{dx} = \frac{y_1 g}{W} = \frac{(e^x)(e^x \ln x)}{(e^{2x})}$

$= \ln x$

Hence $u_2 = x \ln x - x$

$\therefore y_p = u_1 y_1 + u_2 y_2$

$= \left(-\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2\right)(e^x) + (x \ln x - x)(xe^x)$

$= \frac{-3}{4} x^2 e^x + \frac{1}{2} x^2 e^x \ln x$

Then $y(x) = y_c + y_p$

$= C_1 e^x + C_2 x e^x - \frac{3}{4} x^2 e^x + \frac{1}{2} x^2 e^x \ln x$

Note that there is no way we could have 'guessed' the particular solution y_p in this case!