Laplace transforms

$f(t)$ function defined for all $t \geq 0$

the Laplace transform of f is the function

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

defined for all $s \in \mathbb{R}$ for which the integral exists (is convergent).

Notation: often write $\mathcal{L}(f)$ or $\mathcal{L}(f)(s)$ for $F(s)$.

What are they for? Solving linear differential equations

eg: $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cos x$$

we will see how they help with this later.

EXAMPLES

$f(t) = 1$

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}\, dt = \lim_{x \to \infty} \left[-\frac{e^{-st}}{s} \right]_0^x$$

$$= \lim_{x \to \infty} -\frac{e^{-sx}}{s} + \frac{1}{s}$$ limit exists only if $s > 0$

$$= \frac{1}{s}$$

$g(t) = t$ recall: integration by parts $\int_a^b u v' = uv\Big|_a^b - \int_a^b u' v$

$$G(t) = \int_0^\infty t e^{-st}\, dt = \lim_{x \to \infty} \left(-t\frac{e^{-st}}{s}\Big|_0^x + \int_0^x \frac{e^{-st}}{s}\, dt \right)$$

u v'

$v = -\frac{e^{-st}}{s}$

$$= \lim_{x \to \infty} \left(-x\frac{e^{-sx}}{s} + \left[-\frac{e^{-st}}{s^2} \right]_0^x \right)$$

$$= \lim_{x \to \infty} \left(-x\frac{e^{-sx}}{s} - \frac{e^{-sx}}{s^2} + \frac{1}{s^2} \right)$$

$$= 0^{*} + 0 + \frac{1}{s^2} \quad \underline{\text{for } s > 0}$$

$$= \frac{1}{s^2}$$

$$* \quad \lim_{x \to \infty} \frac{-xe^{-sx}}{s} = \lim_{x \to \infty} \frac{-x}{se^{sx}} \underset{\text{L'Hospital's rule}}{=} \lim_{x \to \infty} \frac{-1}{s^2 e^{sx}} = 0$$

Function $f(t)$	Laplace transform $\mathcal{L}(f)(s) = F(s)$
1	$1/s$
t	$1/s^2$

If $F(s)$ is the Laplace transform of $f(t)$, we say that $f(t)$ is the inverse Laplace transform of $F(s)$.

Notation: $f(t) = \mathcal{L}^{-1}(F(s))(t)$

or $f = \mathcal{L}^{-1}(F)$

eg: from above, $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

Laplace transforms – more examples

Consider $f(t) = e^{-at}$, $t \geq 0$ and $a \in \mathbb{R}$ a constant.

$$F(s) = \int_0^\infty e^{-st} e^{-at} dt = \lim_{x \to \infty} \int_0^x e^{-(a+s)t} dt$$

$$= \lim_{x \to \infty} \left[\frac{e^{-(a+s)t}}{a+s} \right]_0^x$$

$$= \lim_{x \to \infty} \left(\frac{e^{-(a+s)x}}{a+s} - \frac{1}{a+s} \right)$$

limit exists only if $a+s > 0$

$s > -a$

$$= 0 - \frac{1}{a+s}$$

$$= \frac{1}{a+s}$$

Consider $g(t) = \sin(at)$, $a \neq 0$ constant

$$G(s) = \int_0^\infty \underbrace{e^{-st}}_{v'} \underbrace{\sin(at)}_{u} dt$$

$v = -\frac{e^{-st}}{s}$

$$= -\frac{e^{-st}}{s} \sin(at) \Big|_0^\infty + \frac{a}{s} \int_0^\infty \underbrace{e^{-st}}_{v'} \underbrace{\cos at}_{u} \, dt$$

$$= -\frac{e^{-st}}{s} \sin(at) \Big|_0^\infty - \frac{a e^{-st}}{s^2} \cos at \Big|_0^\infty + \frac{a^2}{s^2} \int_0^\infty e^{-st} \sin(at) \, dt$$

now $$-\frac{e^{-sx}}{s} \leq \frac{e^{-sx} \sin(ax)}{s} \leq \frac{e^{-sx}}{s}$$ for $s > 0$

$x \to \infty$ $\searrow$ $\downarrow$ $\swarrow$

0 (squeeze theorem)

similarly $\dfrac{e^{-sx}\cos(ax)}{s} \to 0$, and therefore

$$G(s) = \frac{a}{s^2} e^0 \cos(0) + \frac{a^2}{s^2} G(s)$$

rearrange $G(s)\left(1 - \frac{a^2}{s^2}\right) = \frac{a}{s^2}$

$$G(s) = \frac{a}{s^2\left(1 - a^2/s^2\right)} = \frac{a}{s^2+a^2} \quad (s>0)$$

Now we know:

$f(t)$	1	t	e^{-at}	$\sin(at)$
$F(s)$	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s+a}$	$\frac{a}{s^2+a^2}$

Linearity of the Laplace transform

If $f(t)$ and $g(t)$ have Laplace transforms $\mathcal{L}(f)$ and $\mathcal{L}(g)$ which exist for all $s \geq a \in \mathbb{R}$, then for any $\alpha, \beta \in \mathbb{R}$:

$$\mathcal{L}(\alpha f(t) + \beta g(t))(s) = \int_0^\infty e^{-st}(\alpha f(t) + \beta g(t))\,dt$$

$$= \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt$$

$$= \alpha \mathcal{L}(f)(s) + \beta \mathcal{L}(g)(s)$$

i.e. the Laplace transform is a linear transformation (but on an infinite dimensional vector space of functions – not $\mathbb{R}^n$)

It can be shown from this that the inverse Laplace transform is also linear.

EXAMPLES we will use our table

$f(t)$	1	t	e^{-at}	$\sin(at)$
$F(s)$	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s+a}$	$\frac{a}{s^2+a^2}$

Let $f(t) = 2t + e^{5t}$, then

$$F(s) = \mathcal{L}(f)(s) = \mathcal{L}(2t + e^{5t})(s)$$

$$= 2\mathcal{L}(t)(s) + \mathcal{L}(e^{5t})(s)$$

$$= 2\frac{1}{s^2} + \frac{1}{s-5}$$

If $G(s) = \frac{1}{s} - \frac{1}{s^2+2}$ then

$$g(t) = 1 - \sin\sqrt{2}\,t$$

Inverse Laplace transforms of rational functions

Suppose $F(s) = \dfrac{2s-1}{(s^2-1)(s+3)}$, we can't use linearity of $\mathcal{L}^{-1}$ directly – need to write $F(s)$ in partial fractions

$$\frac{2s-1}{(s-1)(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$2s-1 = A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1)$$

sub $s = 1$: $\quad 2-1 = A(2)(4)$

$$A = \frac{1}{8}$$

$s = -1$ $\quad 2(-1)-1 = B(-2)(2)$

$$B = \frac{3}{4}$$

$s = -3$ $\quad -6-1 = C(-4)(2)$

$$C = \frac{7}{8}$$

$$F(s) = \frac{1}{8(s-1)} + \frac{3}{4(s+1)} + \frac{7}{8(s+3)}$$

so

$$f(t) = \frac{1}{8}e^{t} + \frac{3}{4}e^{-t} + \frac{7}{8}e^{3t}$$

Laplace transforms of derivatives

Try to express $\mathcal{L}(f')$ in terms of $\mathcal{L}(f)$: $\int uv' = uv - \int u'v$

$$\mathcal{L}(f') = \int_0^\infty \underset{u}{e^{-st}}\, \underset{v'}{f'(t)}\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$v = f$

$$= \lim_{x\to\infty} e^{-sx} f(x) - f(0) + s\mathcal{L}(f)$$

without more information about f, we can't be sure that the above limit exists.

The standard extra piece of information is to assume that $f(t)$ is bounded by an exponential function, i.e. there exist constants $M > 0$ and $\gamma \in \mathbb{R}$ such that

$$|f(t)| \le M e^{\gamma t} \quad \text{for all} \quad t \ge 0$$

In this case we say f is of exponential order

how does this help?

$$-Me^{\gamma x} \le f(x) \le Me^{\gamma x}$$

$$-Me^{\gamma x} e^{-sx} \le e^{-sx} f(x) \le Me^{\gamma x} e^{-sx}$$

$$-M e^{(\gamma - s)x} \le e^{-sx} f(x) \le M e^{(\gamma - s)x}$$

if $(\gamma - s) < 0$:
$\gamma < s$

$x \to \infty$: $\quad 0$ (squeeze theorem)

and then since $e^{-sx} f(x) \to 0$ we have

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

Example for $f(t) = \sin(at)$ we found $F(s) = \dfrac{a}{s^2+a^2}$

$$|\sin(at)| \leq 1 = e^0$$

so $\sin(at)$ is of exponential order, we can apply the formula

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(a\cos(at)) = s\,\mathcal{L}(\sin(at)) - \sin(0)$$

$$a\mathcal{L}(\cos(at)) = s\,\frac{a}{s^2+a^2}$$

$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$

Second derivatives

$$\mathcal{L}(f'') = \int_0^\infty \underset{u}{e^{-st}}\, \underset{v'}{f''(t)}\, dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)\, dt$$

$$= \lim_{x\to\infty} e^{-sx} f'(x) - f'(0) + s\left(s\mathcal{L}(f) - f(0)\right)$$

$$= s^2\mathcal{L}(f) - s f(0) - f'(0)$$

* assuming f' is of exponential order *

Solving linear ODE with Laplace transforms

Recall: linear ODE

First order: $\frac{dy}{dx} + f(x)y = g(x)$

second order: $\frac{d^2y}{dx} + p(x)\frac{dy}{dx} + q(x)y = g(x)$

Solution methods from MATH1011:

separation of variables, integrating factor (first order ODE)

characteristic equation, variation of parameters, method of undetermined coefficients. (2^{nd} order constant coefficients)

New method: take the Laplace transform of each side of the eqn, re-arrange, take inverse Laplace transform.

EXAMPLE

To solve the initial value problem

$$y''(t) - y(t) = t \quad \text{with} \quad y(0) = 1, \ y'(0) = 1$$

Apply Laplace transform (using table) denoting $Y(s) = \mathcal{L}(y)$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s^2}$$

$$s^2 Y(s) - s - 1 - Y(s) = \frac{1}{s^2}$$

$$(s^2 - 1)Y(s) = \frac{1}{s^2} + s + 1$$

$$Y(s) = \frac{1}{s^2(s^2-1)} + \frac{s+1}{s^2-1}$$

$$= \frac{s^3 + s^2 + 1}{s^2(s-1)(s+1)}$$

partial fraction expansion (omitting working) gives:

$$Y(s) = -\frac{1}{s^2} + \frac{3}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s+1}$$

take inverse Laplace transform

$$y(t) = -t + \frac{3}{2} e^{t} - \frac{1}{2} e^{-t}$$

Check: $y(0) = \frac{3}{2} - \frac{1}{2} = 1$ ✓

$y'(0) = -1 + \frac{3}{2} + \frac{1}{2} = 1$ ✓

Note:

This method only finds solutions of the ODE which actually have Laplace transforms. It is possible to show (using techniques beyond the scope of this course) that if the ODE has constant coefficients then all solutions have Laplace transforms.

Shift theorems

s-shifting

If $F(s)$ is the Laplace transform of $f(t)$ then

$$\mathcal{L}(e^{at} f(t))(s) = F(s-a)$$ = F shifted a units to the right.

Proof

$$\mathcal{L}(e^{at} f(t))(s) = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

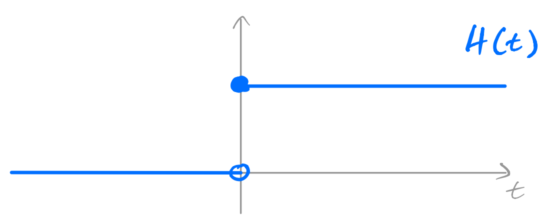
$$= \mathcal{L}(f(t))(s-a)$$

Example $\mathcal{L}(t^2 e^{3t})$

$$\mathcal{L}(t^2) = \frac{2}{s^3} \quad \text{so} \quad \mathcal{L}(t^2 e^{3t}) = \frac{2}{(s-3)^3}$$

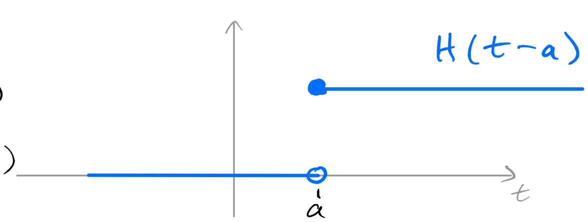
The Heaviside function (unit step function) is defined by

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



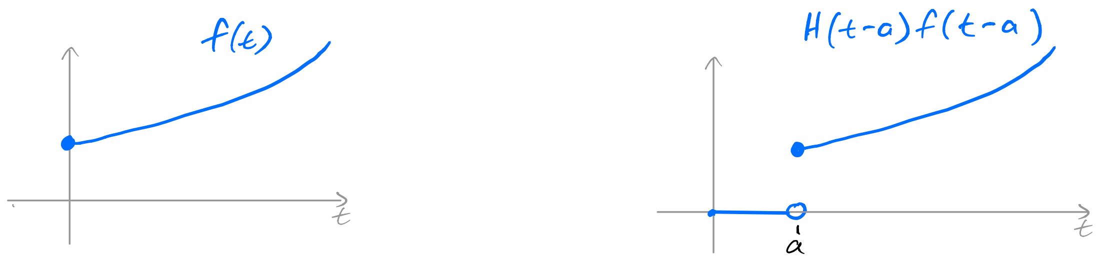
shifted $H(t-a)$, $a \geq 0$:

$$H(t-a) = \begin{cases} 0 & t-a < 0 \quad (t<a) \\ 1 & t-a \geq 0 \quad (t \geq a) \end{cases}$$



t-shifting a function $f(t)$, $t \geq 0$ means not 'activating' it until $t = a$, i.e. using

$$f(t-a)\,H(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$



If $F(s)$ is the Laplace transform of $f(t)$ then for $a \geq 0$

$$\mathcal{L}\left(H(t-a)\, f(t-a) \right) = e^{-as} F(s)$$

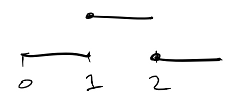
We won't prove this... but it is very useful for finding Laplace transforms for piecewise-defined functions, which are ubiquitous in electrical/electronic engineering applications (eg: flicking a switch on a DC circuit → Heaviside function)

Piecewise defined functions

EXAMPLES

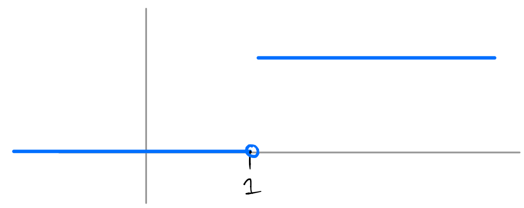
Find $\mathcal{L}(g)$ where $g(t) = \begin{cases} 0 & t < 1 \\ 1 & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$

pulse function

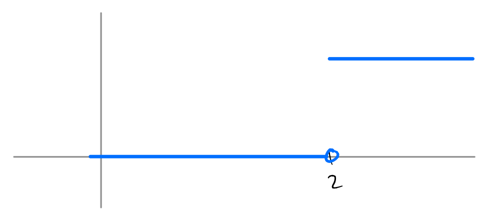


write $g(t)$ in terms of shifted Heaviside.

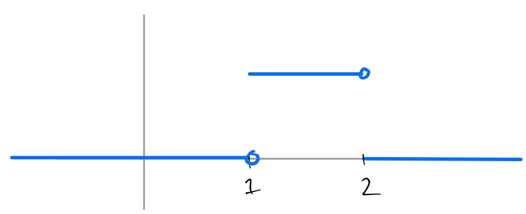
$H(t-1)$ switches on $(=1)$ at $t = 1$



$H(t-2)$ switches on at $t = 2$



so $H(t-1) - H(t-2)$ switches on and then off.



$$g(t) = H(t-1) - H(t-2)$$

$$= f(t-1)\,H(t-1) - f(t-2)\,H(t-2) \quad \text{where } f(t) = 1$$

so

$$\mathcal{L}(g(t)) = \mathcal{L}(f)\,e^{-s} - \mathcal{L}(f)\,e^{-2s}$$

$$= \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$e^{-as}F(s)$	$f(t-a)H(t-a)$

MORE DIFFICULT: Find $\mathcal{L}(g)$ where

$$g(t) = \begin{cases} t & 0 \le t < 3 \\ 1-3t & 3 \le t < 4 \\ 2 & 4 \le t \end{cases}$$

$$g(t) = t\left(H(t) - H(t-3)\right) + (1-3t)\left(H(t-3) - H(t-4)\right)$$
$$+ 2H(t-4)$$

$$= tH(t) + (-t + 1 - 3t)H(t-3) + (-1 + 3t + 2)H(t-4)$$

$$= tH(t) + (1-4t)H(t-3) + (1+3t)H(t-4)$$

we need to get each term in the form $f(t-a)H(t-a)$

first write $1-4t$ as a function of $t-3$:

$$1-4t = 1-4(t-3+3) \qquad (t = t-3+3)$$

$$= 1-4(t-3)-12$$

$$= -4(t-3)-11$$

next $1+3t$ as a function of $t-4$:

$$1+3t = 1+3(t-4+4)$$

$$= 1+3(t-4)+12$$

$$= 3(t-4)+13$$

so

$$g(t) = tH(t) + \left(-4(t-3)-11\right)H(t-3) + \left(3(t-4)+13\right)H(t-4)$$

$f_1(t)H(t)$ $\qquad$ $f_2(t-3)H(t-3)$ $\qquad$ $f_3(t-4)H(t-4)$

$f_1(t) = t$ $\qquad$ $f_2(t) = -4t-11$ $\qquad$ $f_3(t) = 3t+13$

$F_1(s) = \frac{1}{s^2}$ $\qquad$ $F_2(s) = -\frac{4}{s^2} - \frac{11}{s}$ $\qquad$ $F_3(s) = \frac{3}{s^2} + \frac{13}{s}$

using the shift theorem:

$e^{-as}F(s)$	$f(t-a)H(t-a)$

$$G(s) = e^{0}\frac{1}{s^2} + e^{-3t}\left(-\frac{4}{s^2} - \frac{11}{s}\right) + e^{-4t}\left(\frac{3}{s^2} + \frac{13}{s}\right)$$

11.8 Laplace transforms table

$$\mathcal{L}\left(f(t)\right) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

SPECIFIC FUNCTIONS		GENERAL RULES		
$F(s)$	$f(t)$	$F(s)$	$f(t)$	
$\frac{1}{s}$	1	$\frac{e^{-as}}{s}$	$H(t-a)$	
$\frac{1}{s^n}, \quad n \in \mathbb{Z}^+$	$\frac{t^{n-1}}{(n-1)!}$	$e^{-as}F(s)$	$f(t-a)H(t-a)$	t-shift
$\frac{1}{s+a}$	e^{-at}	$F(s-a)$	$e^{at}f(t)$	s-shift
$\frac{1}{(s+a)^n}, \quad n \in Z^+$	$e^{-at}\frac{t^{n-1}}{(n-1)!}$	$sF(s) - f(0)$	$f'(t)$	
$\frac{1}{s^2+\omega^2}$	$\frac{\sin(\omega t)}{\omega}$	$s^2F(s) - sf(0) - f'(0)$	$f''(t)$	
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$	$F'(s)$	$-tf(t)$	
$\frac{1}{(s+a)^2+\omega^2}$	$\frac{e^{-at}\sin(\omega t)}{\omega}$	$F^{(n)}(s)$	$(-t)^n f(t)$	
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos(\omega t)$	$\frac{F(s)}{s}$	$\int_0^t f(u)\,du$	
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{\sin(\omega t) - \omega t\cos(\omega t)}{2\omega^3}$	$F(s)G(s)$	$(f * g)(t)$	
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin(\omega t)}{2\omega}$			

Higher derivatives:

$$\mathcal{L}\left(f^{(n)}(t)\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

The Convolution Theorem:

$$\mathcal{L}(f * g) = \mathcal{L}(f)\,\mathcal{L}(g) \quad \text{where} \quad (f * g)(t) = \int_0^t f(u)g(t-u)\,du$$