Sequences and limits

A **sequence** is an infinite, ordered collection of real numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

notation: $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ (doesn't have to start at $n=1$)

EXAMPLES

$5, 7, 9, 11, \ldots$ $\quad a_n = a_{n-1} + 2, \quad a_1 = 5$

$3, 6, 9, 12, \ldots$ $\quad a_n = a_{n-1} + 3, \quad a_1 = 3$

Arithmetic sequences

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\quad a_n = \frac{1}{n}$ $\quad$ harmonic sequence

$-1, 1, -1, 1, \ldots$ $\quad b_n = (-1)^n$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ $\quad a_n = \left(\frac{1}{2}\right)^{n-1}$ $\quad$ geometric sequence

$1, 1, 2, 3, 5, 8, 13, \ldots$ $\quad$ Fibonacci sequence

$a_1 = a_2 = 1 \qquad a_n = a_{n-1} + a_{n-2}$

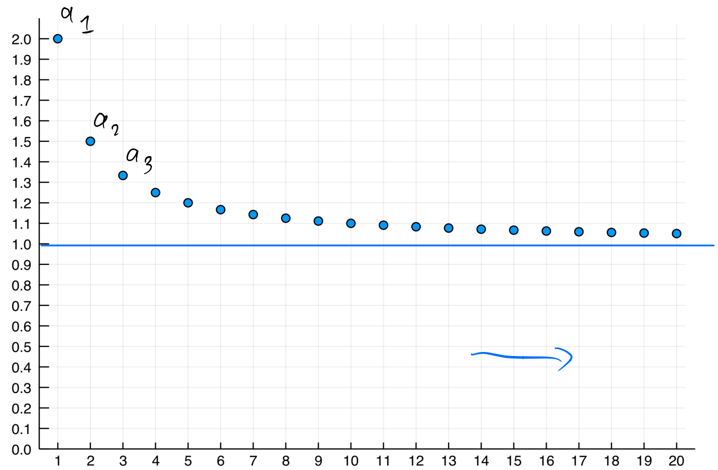
Limits

Definition 8.2. *(Intuitive definition of the limit of a sequence)* *Let $\{a_n\}$ be a sequence and L be a real number. We say that $\{a_n\}$ has a* limit *L if we can make a_n arbitrarily close to L by taking n to be sufficiently large. We denote this situation by*

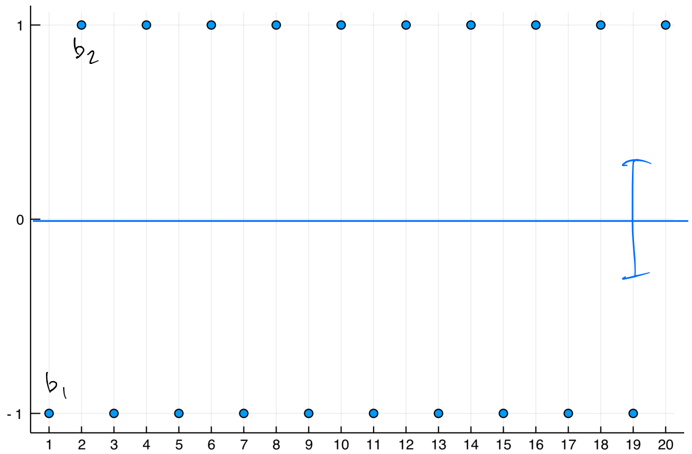
$$\lim_{n\to\infty} a_n = L.$$

We say that $\{a_n\}$ is convergent *if $\lim_{n\to\infty} a_n$ exists; otherwise we say that $\{a_n\}$ is* divergent.

EXAMPLES

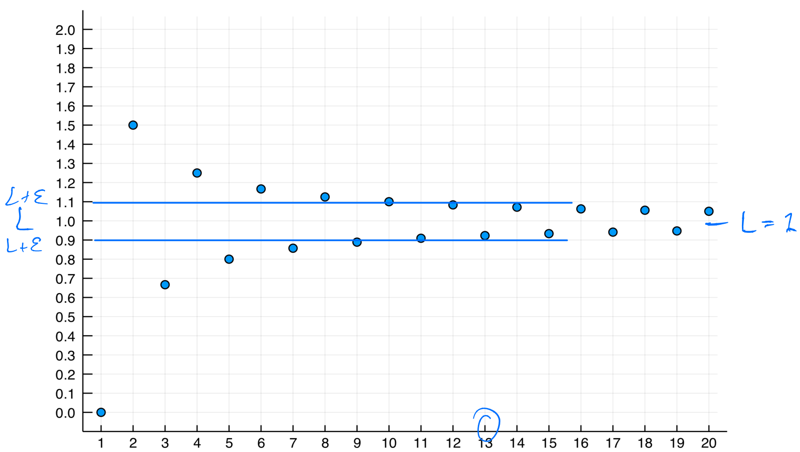


$a_n = 1 + \frac{1}{n}$



$b_n = (-1)^n$

divergent



$a_n = 1 + \frac{(-1)^n}{n}$

$\lim_{n\to\infty} a_n = 1$

The formal definition:

$\lim_{n\to\infty} a_n = L$ if for all $\varepsilon > 0$ there exists N such that if $n > N$

then $|a_n - L| < \varepsilon$

Theorem 8.4 (Limit laws). *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Then:*

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = a \pm b$.

2. $\lim\limits_{n\to\infty} (c\, a_n) = c\, a$ *for any constant* $c \in \mathbb{R}$.

3. $\lim\limits_{n\to\infty} (a_n\, b_n) = a\, b$.

4. *If* $b \neq 0$ *and* $b_n \neq 0$, *for all* n *then* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

eg:

$$\lim_{n\to\infty} \frac{n^2-n+1}{3n^2+2n-1} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}{\cancel{n^2}\left(3+\frac{2}{n}-\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1 - n + \frac{1}{n^2}}{3+\frac{2}{n}-\frac{1}{n^2}}$$

$$= \frac{1-0+0}{3+0-0} = \frac{1}{3}$$

Theorem 8.5 (The squeeze theorem or the sandwich theorem). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = a$ and*

$$a_n \leq b_n \leq c_n$$

for all $n \geq 1$. Then the sequence $\{b_n\}$ is also convergent and $\lim\limits_{n\to\infty} b_n = a$.

$$\begin{array}{ccccc} a_n & \leq & b_n & \leq & c_n \\ \downarrow & & \downarrow & & \downarrow \\ a & & a & & a \end{array}$$

eg: $\lim\limits_{n\to\infty} \dfrac{\cos n}{n}$

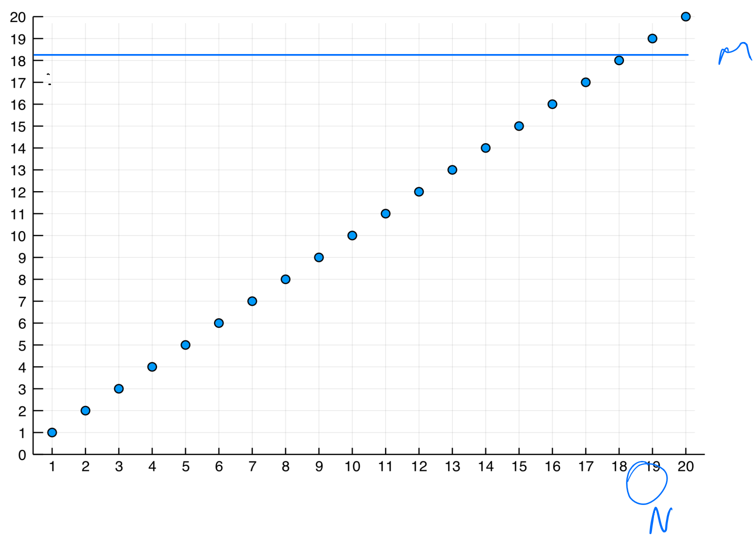
$$\begin{array}{ccccc} -\frac{1}{n} & \leq & \frac{\cos n}{n} & \leq & \frac{1}{n} \\ \downarrow & & & & \downarrow \\ 0 & & & & 0 \end{array}$$

by squeeze theorem $\dfrac{\cos n}{n} \to 0$

Diverging to infinity

We say a sequence a_n diverges to infinity $\left(\lim_{n\to\infty} a_n = \infty \text{ or } a_n \to \infty\right)$ if for any number $M > 0$ there is a term in the sequence after which all terms are bigger than M.

Formally: for all $M > 0$, there exists N such that if $n > N$ the $a_n > M$.



$a_n = n$

$\lim_{n\to\infty} n = \infty$

Similarly, $a_n \to -\infty$ if for all $M < 0$ there exist N such that if $n > N$ then $a_n < M$

Warning: don't treat ∞ like a number / actual limit

if $a_n \to \infty$ and $b_n \to -\infty$ it does NOT follow that $\lim_{n\to\infty} (a_n + b_n) = 0$

eg: $a_n = n \to \infty$

$b_n = -2n \to -\infty$

$a_n + b_n = -n \to -\infty$

Some properties:

If $a_n \neq 0$ for all n, then $a_n \to 0 \iff \frac{1}{|a_n|} \to \infty$

If $a_n > 0$ then $a_n \to \infty \iff \frac{1}{a_n} \to 0$. (1)

Example geometric sequence

$\underline{a_n = r^n}$ if $r > 1$ $\quad r^n \to \infty$ (2)

$r < -1$ $\quad$ divergent

$0 < r < 1$ $\quad \frac{1}{r} > 1$

$$(2) \Rightarrow \left(\frac{1}{r}\right)^n \to \infty \underset{(1)}{\Longleftrightarrow} r^n \to 0$$

The monotone sequences theorem

Bounded sequences

A sequence is called

bounded above if there is a number A such that $a_n \leq A$ for all n

bounded below if there is B s.t. $a_n \geq B$

bounded if it is bounded above and below

eg: $a_n = n$ bounded below by 1, not bounded above

∗ $a_n = \frac{1}{n}$ bounded above by 1, bounded below by 0, bounded

$a_n = n \sin n$

not bounded above
not bounded below.

Every convergent sequence is bounded, but the converse is not true: eg: $(-1)^n$ bounded, divergent

Upper (and lower bounds) are not unique

We call the smallest (least) upper bound the supremum $\sup\{a_n\}$

The greatest lower bound is called the infimum $\inf\{a_n\}$

If there is a maximum then sup = max, if there is a minimum then inf = min

eg: $a_n = \frac{1}{n}$ has $\sup\{a_n\} = 1 = \max\{a_n\}$

$\inf\{a_n\} = 0$, no minimum.

A sequence is called monotone if it is

non-decreasing: $a_1 \leq a_2 \leq a_3 \leq \dots a_n \leq a_{n+1} \dots$ or

non-increasing: $a_1 \geq a_2 \geq a_3 \geq \dots a_n \geq a_{n+1} \dots$

Theorem 8.15 (The monotone sequences theorem). *If the sequence $\{a_n\}_{n=1}^{\infty}$ is non-decreasing and bounded above, then the sequence is convergent and*

$$\lim_{n\to\infty} a_n = \sup(\{a_n\}).$$

If $\{a_n\}_{n=1}^{\infty}$ is non-increasing and bounded below, then $\{a_n\}$ is convergent and

$$\lim_{n\to\infty} a_n = \inf(\{a_n\}).$$

That is, every monotone bounded sequence is convergent.

Infinite series

An infinite series is the sum of the terms in a sequence.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots .$$

For each $n \geq 1$, the n^{th} partial sum is

$$s_n = a_1 + a_2 + \dots + a_n = \sum_{i=1}^{n} a_i$$

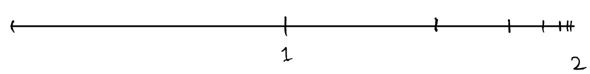
The partial sums of an infinite series form a sequence:

$$s_1, s_2, s_3, \dots, s_n, \dots$$

If this sequence has a limit $\lim_{n\to\infty} s_n = s$, then we say the infinite series is convergent and write $\sum_{n=1}^{\infty} a_n = s$

i.e. $$s = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} (a_1 + a_2 + \dots + a_n)$$

EXAMPLE $$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$$



recall: geometric sequence $(1, r, r^2, \dots, r^n, \dots)$, $r \in \mathbb{R}$

geometric series $$\sum_{i=0}^{\infty} r^i = 1 + r + r^2 + \dots$$

does this series converge?

the n^{th} partial sum is $s_n = 1 + r + r^2 + \dots + r^{n-1}$

$$\Rightarrow \quad r s_n = r + r^2 + r^3 + \dots + r^n$$

$$s_n - r s_n = 1 - r^n$$

$$s_n(1-r) = 1 - r^n$$

$$s_n = \frac{1-r^n}{1-r} \quad \text{for } r \neq 1$$

does the <u>sequence</u> s_n converge? this depends on r:

if $|r|<1$ then $\lim_{n\to\infty} r^n = 0$ so

$$s_n = \frac{1-r^n}{1-r} \xrightarrow[n\to\infty]{} \frac{1-0}{1-r} = \frac{1}{1-r} \qquad \text{(convergent)}$$

$$\left(\text{eg: } r=\tfrac{1}{2},\ \frac{1}{1-\frac{1}{2}} = 2\right)$$

if $r>1$ then $r^n \xrightarrow[n\to\infty]{} \infty$

and since $s_n = 1+r+r^2+\dots+r^n > r^n$

it follows that $s_n \to \infty$ too (divergent)

if $r<1$ then $\lim_{n\to\infty} r^n$ doesn't exist so

$$s_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{1}{1-r} r^n \quad \text{also diverges}$$

if $r=1$ then $s_n = 1+1+\dots+1 = n \to \infty$ divergent

if $r=-1$ then the sequence $(s_n) = (1,0,1,0,1,0,\dots)$ which is divergent.

To summarise: the geometric series is convergent iff $|r|<1$

Theorem 8.27. *If the infinite series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent, then the series* $\sum_{n=1}^{\infty} (a_n \pm b_n)$ *and* $\sum_{n=1}^{\infty} c\, a_n$ *(for any constant* $c \in \mathbb{R}$*) are also convergent with*

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n \quad \text{and} \quad \sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n.$$

Example
$$\sum_{n=0}^{\infty} \left(\left(-\frac{2}{3}\right)^n + \frac{3}{4^{n+1}} \right)$$

$$= \sum_{n=0}^{\infty} \left(\underline{\left(-\frac{2}{3}\right)^n} + \frac{3}{4} \underline{\underline{\left(\frac{1}{4}\right)^n}} \right) \qquad \text{geometric series } \sum r^n \to \frac{1}{1-r}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n + \frac{3}{4} \sum \left(\frac{1}{4}\right)^n$$

$$= \frac{1}{1+2/3} + \frac{3}{4}\,\frac{1}{1-1/4}$$

$$= \frac{3}{5} + \frac{3}{4}\cdot\frac{4}{3} = \frac{8}{5}$$

Divergence test

A bit of logic: If P then Q ← suppose this is true

contrapositive: if not Q then not P ← then so is this

converse: if Q then P ← but this may not be

THEOREM 8.22. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

contrapositive: if $\lim_{n\to\infty} a_n \neq 0$ then the series is not convergent

TRUE "DIVERGENCE TEST"

converse: If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ is convergent

FALSE

Example Is $\sum_{n=1}^{\infty} \frac{n^2-1}{2n^2+2}$ convergent?

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2+2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1-\frac{1}{n^2}\right)}{\cancel{n^2}\,2+2/n^2} = \frac{1-0}{2+0} = \frac{1}{2}$$

therefore divergent by divergence test

Is $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) convergent?

$\lim_{n\to\infty} \frac{1}{n} = 0$ can't decide based only on the divergence test.

(1350 Oresme)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$ divergent