LECTURE 25

Review

sequences $(a_1, a_2, \ldots, a_n, \ldots)$

convergence $\lim_{n\to\infty} a_n = L$

series $a_1 + a_2 + \cdots + a_n + \cdots$

convergence converges if the sequence of partial sums $s_n = \sum_{i=1}^{n} a_i$ has a limit

Divergence test: if a series $\sum_{n=1}^{\infty} a_n$ converges then the terms a_n must converge to zero

i.e. $\sum_{n=1}^{\infty} a_n = s \Rightarrow \lim_{n\to\infty} a_n = 0$

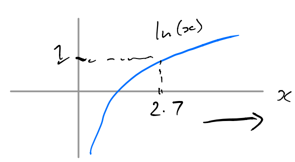
therefore if $\lim_{n\to\infty} a_n \neq 0$, the series $\sum a_n$ must diverge.

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$

Comparison test (CT) $\sum a_n$, $\sum b_n$ series with $0 \le a_n \le b_n$ for all n greater than some fixed N. Then

1. If $\sum b_n$ is convergent then $\sum a_n$ is convergent
2. If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

Example $\sum_{n=1}^{\infty} \underbrace{\frac{\ln(n)}{n}}_{b_n}$ compare with $\sum \underbrace{\frac{1}{n}}_{a_n}$ (harmonic series)



$\ln(x) = 1$
$e^1 = x$
$x \approx 2.7$

for $n \geq 3$ $\quad \ln(n) > 1$

therefore $\quad 0 \leq \frac{1}{n} \leq \frac{\ln(n)}{n} \quad$ for $n \geq 3$

by CT(2), since $\sum \frac{1}{n}$ is divergent, $\sum \frac{\ln(n)}{n}$ is divergent

Limit comparison test (LCT)

$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ series with $a_n \geq 0$ and $b_n \geq 0$ for all n greater than some fixed N. let

$$c = \lim_{n \to \infty} \frac{a_n}{b_n}$$

(a) if $0 < c < \infty$

$$\sum a_n \text{ convergent} \iff \sum b_n \text{ convergent}$$

(b) if $c = 0$

$$\sum b_n \text{ convergent} \implies \sum a_n \text{ convergent}$$

(c) if $\frac{a_n}{b_n} \longrightarrow \infty$

$$\sum a_n \text{ convergent} \implies \sum b_n \text{ convergent}$$

Example $\quad \underbrace{\sum \frac{1}{n^2}}_{a_n}$, $\underbrace{\sum \frac{1}{n^3}}_{b_n}$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{n^2} \div \frac{1}{n^3} = \lim_{n \to \infty} \frac{n^3}{n^2} = \lim_{n \to \infty} n = \infty$$

LCT (c)

$$\sum \frac{1}{n^2} \text{ convergent} \implies \sum \frac{1}{n^3} \text{ convergent}$$

Example $\sum_{n=1}^{\infty} \frac{1+n}{n^2-3}$ convergent?

First try the divergence test:

$$\frac{1+n}{n^2-3} = \frac{\cancel{n^2}\left(\frac{1}{n^2} + \frac{1}{n}\right)}{\cancel{n^2}\left(1 - 3/n^2\right)} \longrightarrow 0$$

LCT? $a_n \geq 0$ ✓ $\underbrace{\frac{1+n}{n^2-3}}_{a_n} \sim \frac{n}{n^2} = \underbrace{\frac{1}{n}}_{b_n}$

$$\frac{a_n}{b_n} = \frac{1+n}{n^2-3} \times n = \frac{\cancel{n^2}\left(\frac{1}{n}+1\right)}{\cancel{n^2}\left(1-3/n^2\right)} \longrightarrow 1$$

LCT (a) $\sum a_n$ convergent $\iff \sum b_n$ convergent

$\sum \frac{1}{n}$ divergent, therefore by LCT (a) $\sum \frac{1+n}{n^2-3}$ divergent

Example $\sum_{n=1}^{\infty} \underbrace{\frac{\ln(n)}{n}}_{a_n}$ (again) compare with $\underbrace{\frac{1}{n}}_{b_n}$

$$\frac{a_n}{b_n} = \frac{\ln(n)}{n} \times n = \ln(n) \underset{n\to\infty}{\longrightarrow} \infty$$

LCT (c): if $\sum \frac{\ln(n)}{n}$ convergent then $\sum \frac{1}{n}$ is convergent

but $\sum \frac{1}{n}$ is divergent! therefore $\sum \frac{\ln(n)}{n}$ is divergent.

Example $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$ compare with $\frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{1+\sin n}{n^2} \times n^2 = 1 + \sin n \longrightarrow ?$$

LCT not applicable

(comparison test works though - see previous lecture)

Example $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n + n}{2n^2 - 1}$ convergent?

div test: $\displaystyle\frac{\sin^2 n + n}{2n^2+1} = \frac{\cancel{n^2}\left(\frac{\sin^2 n}{n^2} + \frac{1}{n}\right)}{\cancel{n^2}(2 + 1/n^2)} \to 0$

$$\underset{n\to\infty}{\sim} \frac{n}{n^2} = \frac{1}{n} \quad \text{compare with } \sum \frac{1}{n}$$

$$\frac{a_n}{b_n} = \frac{(\sin^2 n + n)\, n}{2n^2 - 1} = \frac{n \sin^2 n + n^2}{2n^2 - 1} = \frac{\cancel{n^2}\left(\frac{\sin^2 n}{n} + 1\right)}{\cancel{n^2}\left(2 - 1/n^2\right)}$$

$$0 \le \frac{\sin^2 n}{n} \le \frac{1}{n} \qquad \frac{\sin^2 n}{n} \to 0 \text{ by squeeze theorem}$$

$n \to \infty$: 0

$$\frac{a_n}{b_n} \to \frac{1}{2}$$

LCT (a) $\quad \sum a_n$ convergent $\Leftrightarrow$ $\sum b_n$ convergent

$\sum b_n = \sum \frac{1}{n}$ is divergent, therefore $\displaystyle\sum \frac{\sin^2 n + n}{2n^2 - 1}$ divergent

Example $\displaystyle\sum_{n=1}^{\infty} \frac{2\sqrt{n} + 3}{3n^2 - 1} \sim \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} \qquad \sum b_n = \sum \frac{1}{n^{3/2}}$

$$\frac{a_n}{b_n} = \frac{2\sqrt{n} + 3}{3n^2 - 1} \cdot n^{3/2} = \frac{2n^2 + 3n^{3/2}}{3n^2 - 1} = \frac{n^2}{n^2}\, \frac{\left(2 + 3/\sqrt{n}\right)}{3 - 1/n^2}$$

$$\to \frac{2}{3}$$

LCT (a) $\quad \sum a_n$ convergent $\Leftrightarrow$ $\sum b_n$ convergent

$\sum \frac{1}{n^{3/2}}$ p-series $p > 1$ convergent

therefore $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n + n}{2n^2 - 1}$ is also convergent.

LECTURE 27

Given a series $\sum_{n=1}^{\infty} a_n$, there are two associated sequences

- terms in the series $(a_1, a_2, \dots, a_n, \dots)$
- partial sums $(s_1, s_2, \dots, s_n, \dots)$

$$s_n = \sum_{i=1}^{n} a_i$$

A sequence is actually a function $\mathbb{Z}^+ \to \mathbb{R}$ $\quad n \mapsto a_n$

positive integers (or a subset thereof)

$\mathbb{Z}^+ \subset \mathbb{R}$, and sometimes for a sequence $(a_n)_1^\infty$ there is a continuous function $f : [1,\infty) \to \mathbb{R}$ such that

$$f(1) = a_1, \quad f(2) = a_2, \dots \quad f(n) = a_n \dots$$

(i.e. the sequence is the restriction of f to $\mathbb{Z}^+$)

eg: $\quad a_n = \frac{1}{n}$, $\quad f(x) = \frac{1}{x}$ $\quad$ then $f(n) = a_n$

in such a case it is possible to get information about convergence of the associated series $\sum_{n=1}^{\infty} a_n$ from the improper integral $\int_1^\infty f(x)dx$

more precisely: the integral test

If $\sum_{n=1}^{\infty} a_n$ is a series with each $a_n > 0$, and f a function which is:

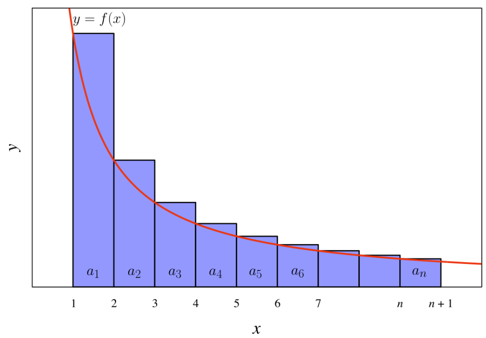
positive $f(x) > 0$, decreasing, and continuous $[1,\infty)$

and satisfies $f(n) = a_n$ for each $n \in \mathbb{Z}^+$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^\infty f(x)\,dx$$

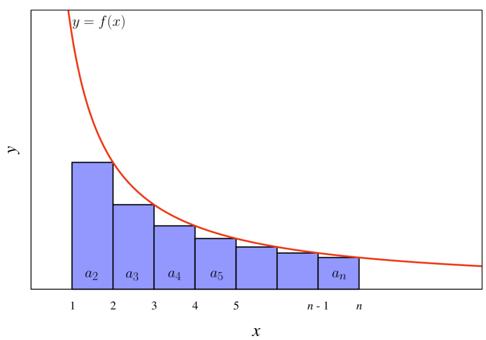
either both converge or both diverge.

<u>proof</u> :



because f is decreasing (assumption):

$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \dots + a_n \qquad (1)$$
$$= s_n$$



also :

$$a_2 + a_3 + \dots + a_n \le \int_1^n f(x)\,dx$$
$$\parallel$$
$$s_n - a_1$$

so $\quad s_n \le \int_1^n f(x)\,dx + a_1 \qquad (2)$

suppose $\int_1^\infty f(x)\,dx = b \in \mathbb{R}$ (i.e. convergent)

then by (2) $\quad s_n \le b + a_1$

because $\int_1^n f(x)\,dx$ is increasing ($f(x) > 0$ by assumption)

so s_n is bounded above (by $b + a_1$)

and s_n is increasing ($a_n > 0$)

therefore by the monotone sequences theorem s_n converges, which means $\sum_{n=1}^\infty a_n$ converges.

If $\int_1^\infty f(x)\,dx$ is divergent, then by (1) s_n has no upper bound, s_n is increasing, s_n diverges $\Rightarrow \sum_{n=1}^\infty a_n$ diverges

$\square$

Recall :

claimed p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow p>1$

divergence test : if $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges

the improper integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges iff $p>1$

(i.e. if $p \leq 1$ it diverges)

convergence of p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p \in \mathbb{R}$.

- if $p \leq 0$ $\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p} \neq 0$ so by divergence test $\sum \frac{1}{n^p}$ diverges

- if $p>0$ then consider $f(x) = \frac{1}{x^p}$, this function is positive, continuous on $[1,\infty)$ and decreasing $\left(f'(x) = \frac{-p}{x^{p+1}} < 0 \right)$
  $f(n) = \frac{1}{n^p}$

i.e. f satisfies the conditions for the integral test, so

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \Leftrightarrow \int_1^{\infty} \frac{1}{x^p} \text{ converges} \Leftrightarrow p>1$$

(above)

Therefore : p-series converges $\Leftrightarrow p>1$

Example Does $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge?

let $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$

then $f(x) > 0$ ✓

f is continuous on $[2, \infty)$

f decreasing? $f(x) = (x \ln x)^{-1}$

$$f'(x) = -1 (x \ln x)^{-2} (1 . \ln x + x . \tfrac{1}{x})$$

$$= -1 \frac{(\ln x + 1)}{(x \ln x)^2} < 0$$

therefore decreasing.

f satisfies the conditions for the integral test.

so: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges iff $\int_2^{\infty} \frac{1}{x \ln x} \, dx$ converges

$$\int_2^t \frac{1}{x \ln x} \, dx = \Big[\ln (\ln x) \Big]_2^t = \ln (\ln t) - \ln (\ln 2) \xrightarrow[t \to \infty]{} \infty$$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)} \quad \text{so} \quad \frac{d}{dx} \ln (\ln x) = \frac{1}{x} \div \ln x = \frac{1}{x \ln x}$$

$\int_2^{\infty} \frac{1}{x \ln x}$ diverges, so $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ also diverges by the integral test.

Lecture 28

$$(-1)^n = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

Alternating series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 \cdots$$

where $a_n \geq 0$

the alternating series test:

let $\{a_n\}$ be a sequence such that for all $n > N$, fixed $N \in \mathbb{Z}$

$$a_n \geq 0$$

$$a_n \geq a_{n+1} \quad \text{(non-increasing)}$$

and such that $\lim\limits_{n\to\infty} a_n = 0$. Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad \text{is convergent.}$$

Example $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$

$\frac{1}{n} \geq 0 \;\; \forall n \geq 1$, $\frac{1}{n} \geq \frac{1}{n+1}$, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. so by the alt. series test $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent

Note: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots = - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

is also convergent.

Absolute convergence

An infinite series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem 8.47 If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is also convergent.

Proof

$$0 \le a_i + |a_i| \le 2|a_i|$$

$$\Rightarrow \quad 0 \le \underbrace{\sum_{i=1}^{n} (a_i + |a_i|)}_{b_n} \le 2\sum_{i=1}^{n} |a_i| \le 2\sum_{i=1}^{\infty} |a_i| = S$$

b_n is a non-decreasing (because $0 \le a_i + |a_i|$) sequence which is bounded above by $2\sum_{n=1}^{\infty} |a_n|$. It therefore converges by the monotone sequences theorem, i.e. $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. But then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|)$$

$$= \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

Convergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

Therefore $\sum_{n=1}^{\infty} a_n$ converges. $\square$

Note: The converse is not true:

convergence does not imply absolute convergence

eg: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges (see above) $\quad \sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic divergent)

Such a series is called conditionally convergent.

Example $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

recall : comparison test

Theorem 8.29 (The comparison test). *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be infinite series such that*

$$0 \le a_n \le b_n$$

holds for all sufficiently large n.

1. *If $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.*
2. *If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ is divergent.*

we can't apply this test to $\frac{\sin n}{n^2}$ because it takes negative values. However

$$0 \le \left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p>1$) so $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges by the comparison test. i.e: $\sum \frac{\sin n}{n^2}$ is absolutely convergent therefore convergent

Theorem 8.50 The ratio test

Suppose $\sum_{n=1}^{\infty} a_n$ is such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \in \mathbb{R}$, then

if $L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

if $L > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: if $L = 1$ the ratio test is inconclusive

Intuition: recall geometric series $\sum_{n=1}^{\infty} r^n$, $r \in \mathbb{R}$, converges iff $|r|<1$

$a_n = r^n$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{r^{n+1}}{r^n} = r$

we can think of the ratio test as saying that if $\left|\frac{a_{n+1}}{a_n}\right| \to L \neq 1$ then $\sum a_n$ behaves like the geometric series with $r = L$

Examples

$$\sum_{n=1}^{\infty} \frac{n}{e^n} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{e^{n+1}} \div \frac{n}{e^n} = \frac{n+1}{e^{n+1}} \times \frac{e^n}{n} = \frac{1}{e}\,\frac{n+1}{n}$$

$$= \frac{1}{e}\,\frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}} \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$$

by the ratio test, this series converges

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n} = 2\cdot\frac{n^2}{n^2+2n+1} = \frac{2}{1+\frac{2}{n}+\frac{1}{n^2}} \longrightarrow 2$$

$2 > 1$ by the ratio test: divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}}{n+1}\cdot\frac{n}{(-1)^n}\right| = \left|\frac{(-1)\,n}{n+1}\right| = \frac{1}{1+\frac{1}{n}} \xrightarrow[n\to\infty]{} 1$$

ratio test is inconclusive

$$\sum_{n=1}^{\infty} \frac{b^n}{n!}, \quad b \in \mathbb{R}$$ exercise

Lecture 29

The ratio test

Suppose $\sum_{n=1}^{\infty} a_n$ is such that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \in \mathbb{R}$, then

if $L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

if $L > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

if $L = 1$ the ratio test is inconclusive

Example

Find all $x \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+3)^n$$

is absolutely convergent / conditionally convergent / divergent.

$x = -3$: $\quad \sum 0 = 0 \quad$ convergent

suppose $x \neq -3$, then applying the ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|(x+3)^{n+1}|}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|(x+3)^n|} = |x+3| \frac{\sqrt{n}}{\sqrt{n+1}} = |x+3| \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}} \frac{\sqrt{1}}{\sqrt{1+\frac{1}{n}}} \xrightarrow[n\to\infty]{} |x+3|$$

if $|x+3| < 1$ abs. convergent

$-1 < x+3 < 1$

$-4 < x < -2$

if $|x+3| > 1$ divergent

$x+3 > 1$ or $x+3 < -1$

$x > -2 \qquad x < -4$

if $|x+3| = 1$ ratio test inconclusive

$x+3 = 1$ or $x+3 = -1$

$x = -2$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

converges by the alternating series test

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \frac{1}{\sqrt{n}}$ p series with $p < 1$ diverges

so when $x = -2$ the series is conditionally convergent.

$x = -4$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (-1)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

divergent (p-series)

summary:

$|x+3| < 1$ converges

$|x+3| > 1$ divergent

$|x+3 = 1$ → $x = -2$ conditionally convergent

→ $x = -4$ divergent

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+3)^n$ → general form: $\sum_{n=1}^{\infty} a_n (x-c)^n$

A power series centred at c is a series of the form:

$$\sum_{n=1}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \dots$$

For any power series there are three possibilities:

1. The series is absolutely convergent when $x = c$ and divergent elsewhere
2. The series is absolutely convergent for all $x \in \mathbb{R}$.
3. There exists $R > 0$ such that the series is absolutely convergent for $|x-c| < R$ and divergent for $|x-c| > R$.

R is called the radius of convergence

Convergence at $|x-c| = R$ depends on the specific (a_n)'s

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+3)^n$$ is a power series centred at -3

it was shown above that this series is absolutely convergent when $|x+3| < 1$, so the radius of convergence is 1.

A power series representation centred at c of a given function f is a power series $\sum_{n=1}^{\infty} a_n (x-c)^n$ such that

$$f(x) = \sum_{n=1}^{\infty} a_n (x-c)^n \quad \text{for } x \in I$$

where I is some interval containing c

(not necessarily the whole domain of f)

Example

Recall the geometric series $1 + r + r^2 + \dots + r^n + \dots$ converges to $\frac{1}{1-r}$ when $|r| < 1$.

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \qquad |x| < 1$$

which is a power series representation for $f(x) = \frac{1}{1-x}$ on the interval $(-1,1)$.

$|x| < 1$

Taylor series

Suppose $f(x)$ has a power series representation at c

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \dots + a_n(x-c)^n + \dots$$

then

$$f(c) = a_0$$

differentiating:

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \dots$$

$$f'(c) = a_1$$

differentiating again:

$$f''(x) = 2a_2 + 3.2a_3(x-c) + 4\times 3a_4(x-c)^2 + \dots$$

$$f''(c) = 2a_2$$

again:

$$f^{(3)}(x) = 3.2a_3 + 4.3.2.a_4(x-c) + \dots$$

$$f^{(3)}(c) = 3.2a_3$$

$\dots$ $f^{(4)}(c) = 4.3.2\,a_4$ $\qquad$ $f^{(n)}(c) = n!\,a_n$

so $\quad a_0 = f(c) \quad a_1 = f'(c) \quad a_2 = \frac{f''(c)}{2} \quad a_3 = \frac{f^{(3)}(c)}{3!}$

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and we have the Taylor series

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

If you have taken MATH 1011 you might remember that the nth-order Taylor polynomial of a function, which is just the partial sum S_{n+1} of the Taylor series, gives an approximation for the function:

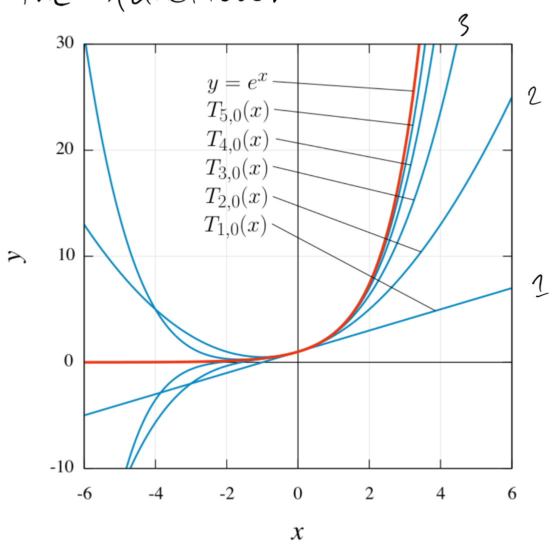


Figure 5.1: The first 5 Taylor polynomials of e^x.

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

the Taylor series (centred at 0) for e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

for which values of x does this series actually converge to e^x? (what is the radius of convergence?)

$$\left| \frac{x^{n+1}}{(n+1)!} \div \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0 \qquad (\text{for all } x \in \mathbb{R})$$

by the ratio test the Taylor series for e^x is absolutely convergent for every $x \in \mathbb{R}$!

another example:

$$f(x) = \ln(1+x) \qquad f(0) = 0 = a_0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1 = a_1$$

$$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1 = a_2$$

$$f^{(3)}(x) = \frac{2}{(1+x)^3} \qquad f^{(3)}(0) = 2 = a_3$$

Taylor series centred at 0:

$$0 + x - \frac{1}{2}x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1}\frac{x^n}{n} + \dots$$

convergence? ratio test:

$$\left| \frac{(-1)^n x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n-1} x^n} \right| = |x| \frac{n}{n+1} = |x| \frac{\cancel{n}}{\cancel{n}} \frac{1}{1+1/n} \xrightarrow[n\to\infty]{} |x|$$

absolutely convergent if $|x| < 1$

divergent if $|x| > 1$

$x = -1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = \sum \frac{(-1)^{2n}(-1)^{-1}}{n} = \sum \frac{(-1)^{-1}}{n} = -\sum \frac{1}{n}$

divergent (harmonic series)

$x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by alternating series test.

$$\ln(x+1) = \ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Moral: the Taylor series at 0 of $\ln(1+x)$ only converges when $x \in (-1, 1]$, it shouldn't be used as an approx. for other values of x.

(will need to use the series centred near the value(s) of interest instead)