Lecture 30

Previous lecture: power series (in particular Taylor series) give a way of expressing a function $f(x)$ as an infinite sum of polynomials (at least within the radius of convergence).

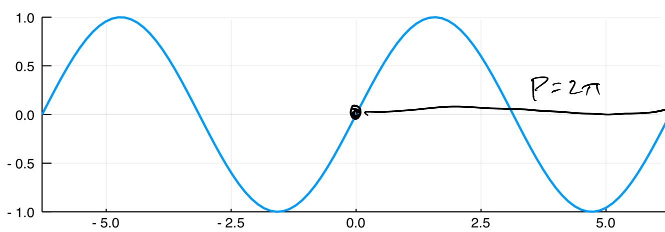
$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

Suppose $f(t)$ is periodic, meaning there is a P such that $f(t+P) = f(t)$ for all t (i.e. f repeats itself after a period P),

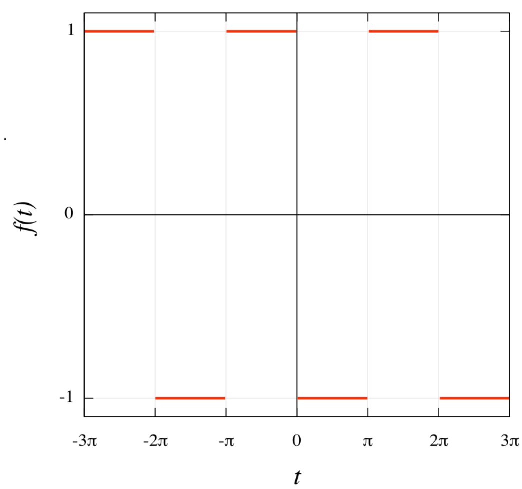
eg:

$f(t) = \sin t$

$= \sin(t+2\pi)$



$$f(t) = \begin{cases} 1, & -\pi < t \leq 0, \\ -1, & 0 < t \leq \pi, \end{cases} \quad \textit{and} \quad f(t+2\pi) = .$$



For 2π periodic functions there is an alternative series representation which is particularly useful called the Fourier series (FS):

$$\frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \dots$$
$$+ b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \dots \qquad a_i, b_i \in \mathbb{R}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

i.e. a Fourier series is just an infinite linear combination of functions in the infinite set

$$\left\{ \frac{1}{2},\ \cos t,\ \sin t,\ \cos 2t,\ \sin 2t,\ \cos 3t,\ \sin 3t,\ \ldots \right\} = B$$

It turns out that every 2π periodic function has a Fourier series representation/expansion which converges to the function (except at points of discontinuity).

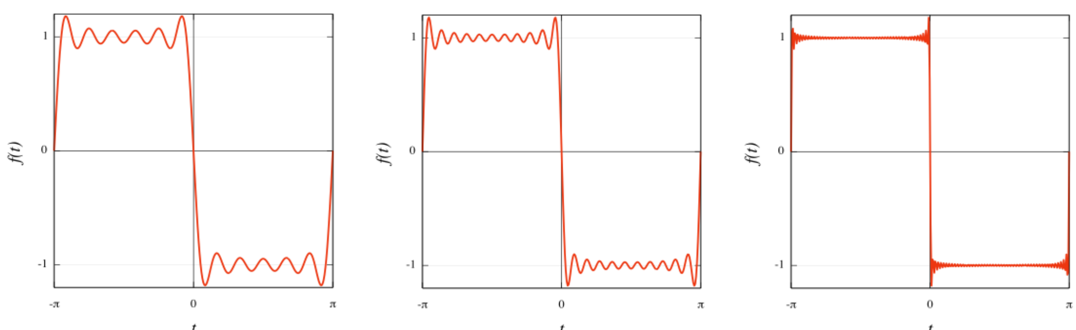
eg:

$$f(t) = \begin{cases} 1, & -\pi < t \leq 0, \\ -1, & 0 < t \leq \pi, \end{cases} \qquad \textit{and} \qquad f(t+2\pi) = f(t).$$

has Fourier series

$$-\frac{4}{\pi} \sum_{\substack{n=1, \\ n \text{ odd}}}^{\infty} \frac{\sin nt}{n}$$

the partial sums look like:



$\sum_{n=1}^{11}$ $\qquad \sum^{21}$ $\qquad \sum^{101}$

Gibb's phenomenon: convergence is not as good near points of discontinuity of $f(t)$.

Finding the coefficients

The constants a_i, b_i in a Fourier series are called the **Fourier coefficients**. How can we find the coefficients for a given $f(t)$?

Recall: for power series we assumed

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

then $f(c) = a_0$, then differentiate $f'(c) = a_1$

ended up with $a_n = \dfrac{f^{(n)}(c)}{n!} \rightarrow$ Taylor series

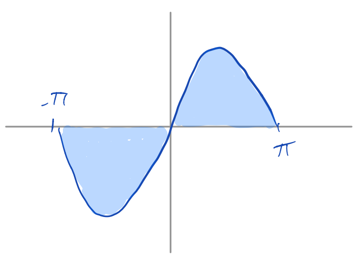
To find Fourier coefficients the basic idea is the same: find a_i, b_i in terms of f by eliminating the other terms in the series.

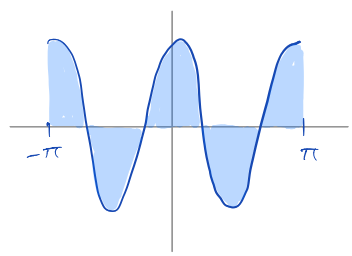
First assume $f(t)$ has a FS representation

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (1)$$

to find a_0: note that $\int_{-\pi}^{\pi} \cos(nt) = 0 \qquad \int_{-\pi}^{\pi} \sin(nt) = 0$

eg:





integrate $\int_{-\pi}^{\pi}$ each side of (1)

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dt + \sum_{n=1}^{\infty}\Big(a_n \overbrace{\cancel{\int_{-\pi}^{\pi} \cos nt\,dt}}^{0} + b_n \overbrace{\cancel{\int_{-\pi}^{\pi} \sin nt\,dt}}^{0}\Big)$$

$$= \left[\frac{a_0}{2} t\right]_{-\pi}^{\pi} + 0$$

$$= \frac{a_0}{2}(\pi - (-\pi)) = \pi a_0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt$$

to find a_n for $n \geq 1$, we will need:

1. $\int_{-\pi}^{\pi} \cos(mt)\sin(nt)\,dt = 0$ $\quad$ m, n integers $\quad m \neq n$

2. $\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = 0$ $\quad$ m, n integers $\quad m \neq n$

3. $\int_{-\pi}^{\pi} \cos^2(mt)\,dt = \pi$

$\cos x = \cos(-x)$

$\cos m\pi \; \frac{\cos n\pi}{n}$

$- \cos(-m\pi) \frac{\cos(-n\pi)}{n}$

I.P. $\int uv' = uv - \int u'v$

proof of 1.

Let $I = \int_{-\pi}^{\pi} \underset{u}{\cos(mt)} \underset{v'}{\sin(nt)}\,dt = \overset{0}{\cancel{-\left[\cos mt \frac{\cos nt}{n}\right]_{-\pi}^{\pi}}} - \int_{-\pi}^{\pi} m \sin mt \frac{\cos nt}{n}\,dt$

$v = -\frac{\cos nt}{n}$

$= -\int_{-\pi}^{\pi} \underset{u}{m \sin mt} \underset{v'}{\frac{\cos nt}{n}}\,dt = \overset{0}{\cancel{-\left[m \sin mt \frac{\sin nt}{n^2}\right]_{-\pi}^{\pi}}} + \int_{-\pi}^{\pi} m^2 \cos mt \frac{\sin nt}{n^2}$

$v = \frac{\sin nt}{n^2}$

$= \frac{m^2}{n^2} \int_{-\pi}^{\pi} \cos mt \sin nt\,dt = \frac{m^2}{n^2} I$

$I - \frac{m^2}{n^2} I = 0 \qquad I\left(1 - \frac{m^2}{n^2}\right) = 0$

$I = 0 \quad$ or $\quad 1 = \frac{m^2}{n^2} \Rightarrow m = n$

since $m \neq n$, $\quad I = 0!$

proof of 2. is similar

proof of 3. half angle formula.

back to FS. $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ (1)

multiply each side of (1) by $\cos mt$ and then $\int_{-\pi}^{\pi}$

$$\int_{-\pi}^{\pi} \cos m t \, f(t)\,dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mt \, dt \; = 0$$

$$+ \sum \int_{-\pi}^{\pi} a_n \cos mt \cos nt \, dt \; = 0 \text{ except when } m = n$$

$$+ \sum \int_{-\pi}^{\pi} b_n \cos mt \sin nt \, dt \; = 0 \text{ except } m = n$$

$$= a_m \int_{-\pi}^{\pi} \cos mt \cos mt \, dt \Big]$$

$$+ b_m \int_{-\pi}^{\pi} \cos mt \sin mt \, dt = b_m \int_{-\pi}^{\pi} \frac{\sin 2mt}{2} dt = 0$$

$$= a_m \int_{-\pi}^{\pi} \cos^2 mt \, dt = a_m \pi$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos mt \, f(t)\,dt$$

a similar argument gives

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin mt \, dt$$

Summary: the FS representation of a 2π-periodic function $f(t)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

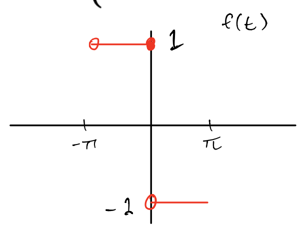
where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\,dt \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nt \, f(t)\,dt$

and $\qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$

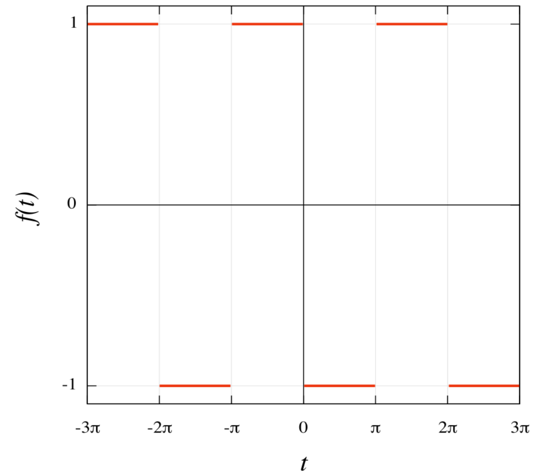
LECTURE 31

Example:

$$f(t) = \begin{cases} 1, & -\pi < t \le 0, \\ -1, & 0 < t \le \pi, \end{cases} \quad \text{and} \quad \underline{f(t+2\pi) = f(t).}$$



periodic extension



finding the Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1\, dt + \int_{0}^{\pi} -1\, dt \right)$$

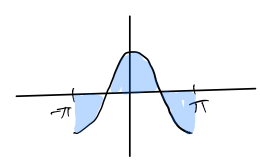
$$= \frac{1}{\pi}\left([t]_{-\pi}^{0} + [-t]_{0}^{\pi} \right)$$

$$= \frac{1}{\pi}\left(0 - (-\pi) + (-\pi) - 0 \right)$$

$$= 0$$

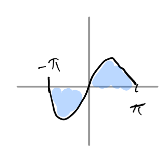
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1 \cos nt\, dt + \int_{0}^{\pi} (-1) \cos nt\, dt \right)$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} \cos nt\, dt - \int_{0}^{\pi} \cos nt\, dt \right)$$



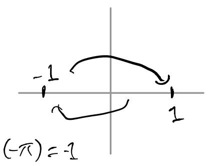
$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt = \frac{1}{\pi}\left(\int_{-\pi}^{0} \sin nt\, dt - \int_{0}^{\pi} \sin nt\, dt \right)$$



$$= \frac{2}{\pi}\left(\int_{-\pi}^{0} \sin nt \right) = \frac{2}{\pi}\left[-\frac{\cos nt}{n} \right]_{-\pi}^{0}$$

$$= \frac{2}{\pi}\left[-\frac{\cos 0}{n} + \frac{\cos(-n\pi)}{n} \right]$$



$\cos(-\pi) = -1$

$\cos(-2\pi) = 1$

$\cos(-3\pi) = -1$

$$= \frac{2}{\pi}\left[\frac{(-1)}{n} + \frac{(-1)^n}{n} \right]$$

$$= \begin{cases} -\dfrac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

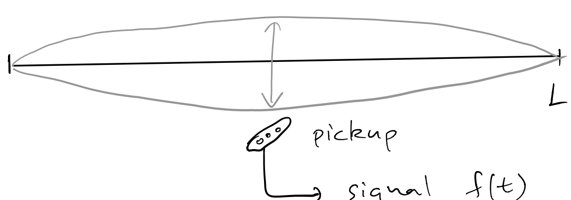
so $$f(t) = -\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin nt = -\sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin (2n-1)t}{2n-1}$$

Fourier Series for functions with arbitrary period.

So far only discussed FS of 2π-periodic functions, but it is not difficult to extend to functions with period $2L$.

Example

Vibrational modes of a guitar string of length L

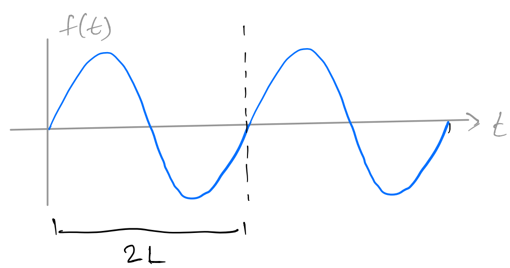


wavelength $\lambda = 2L$

$\lambda = vT$ $\qquad T$ = period

assume $v = 1$

then period $= 2L$

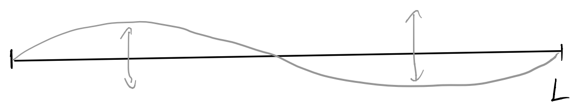


$f(t) = \sin \frac{\pi}{L} t$ $\qquad$ check: $\sin \frac{\pi}{L}(2L) = \sin 2\pi = \sin 0$

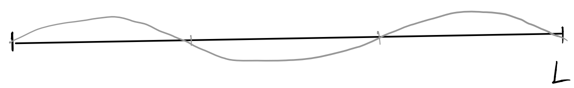
i.e. $f(2L) = f(0)$

guitars don't sound like sine waves...

harmonics:



$\lambda = L \rightarrow \sin \frac{2\pi}{L} t$



$\lambda = \frac{2L}{3} \rightarrow \sin \frac{3\pi}{L} t$

• • •

the actual sound/signal will be a linear combination (superposition) of harmonics (and possibly other noises)

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} t$$

i.e. a Fourier series!

The Fourier coefficients correspond to amplitudes of the harmonics

This is why finding the Fourier coefficients for a function $f(t)$ is sometimes called harmonic analysis

The unique mix of these amplitudes is what gives a signal its characteristic sound.

Fourier series for functions with period 2L

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

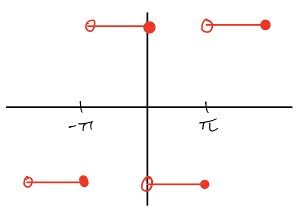
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Convergence of Fourier Series

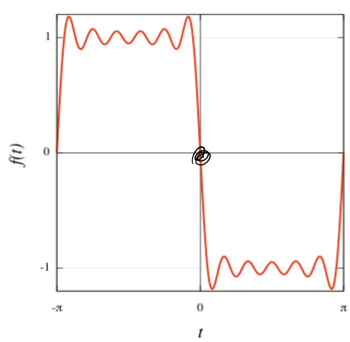
Theorem 9.5

If $f(t)$ and $f'(t)$ are bounded and piecewise continuous on $[-L, L]$ then the FS converges to $f(t)$ except at the $t \in [-L, L]$ at which f is not continuous. At these points of discontinuity the Fourier series converges to the average of the left and right limits.

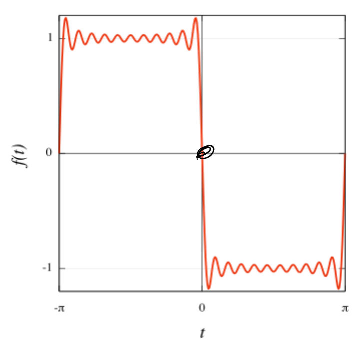
Example



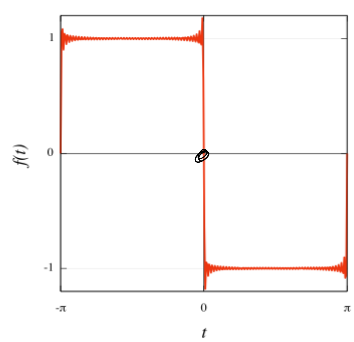
$$f(t) = \begin{cases} 1, & -\pi < t \le 0, \\ -1, & 0 < t \le \pi, \end{cases} \quad \text{and} \quad f(t + 2\pi) = f(t).$$



$\sum^{11}$

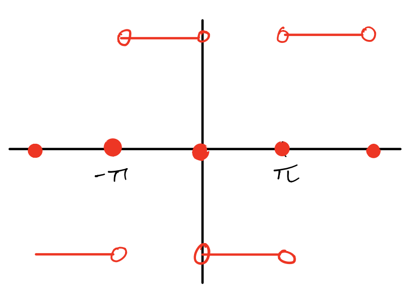


$\sum^{21}$

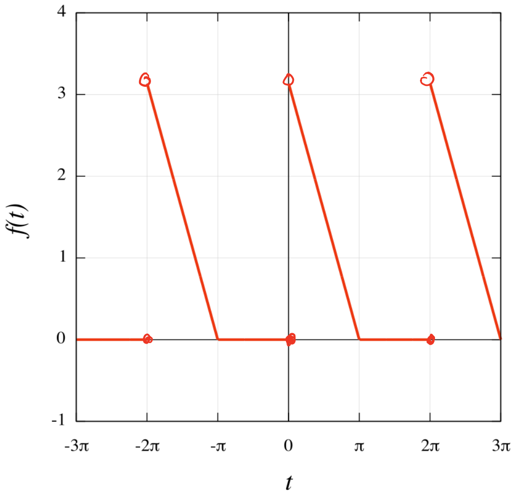


$\sum^{101}$

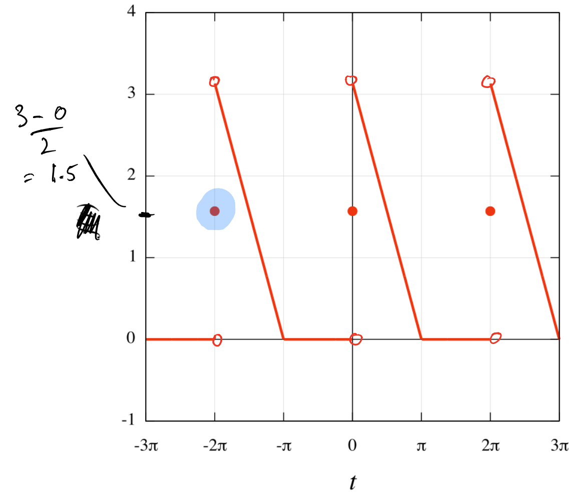
$\sum_{i=1}^{n} \xrightarrow[n \to \infty]{}$



function $f(t)$



Fourier series of $f(t)$



Lecture 32

Let $f:(-L,L] \to \mathbb{R}$, then the periodic extension of $f(t)$ is the function defined by

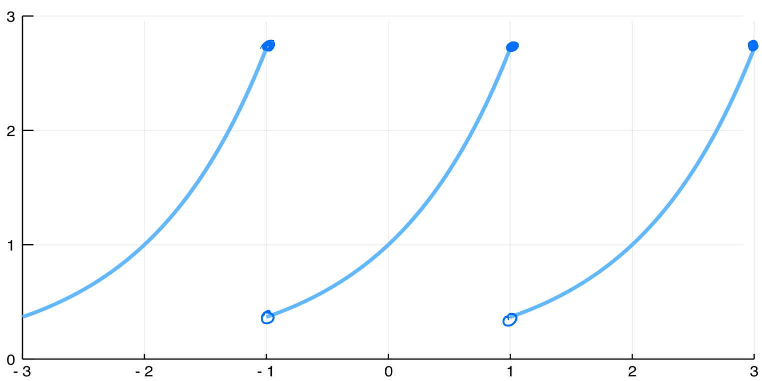
$$\phi(t) = f(t) \quad \text{for} \quad -L < t \leq L$$

and $\phi(t+2L) = \phi(t)$

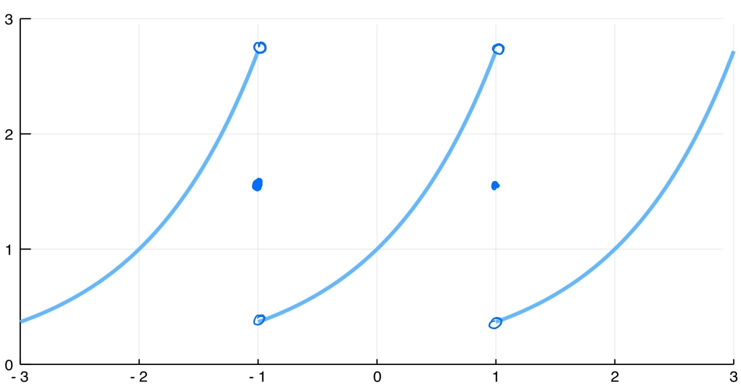
Example

$\phi(t) = e^t$ for $-1 < t \leq 1$ with $\phi(t+2) = \phi(t)$

period $= 2$



$\phi(t)$



Fourier series of $\phi(t)$

Odd and even functions

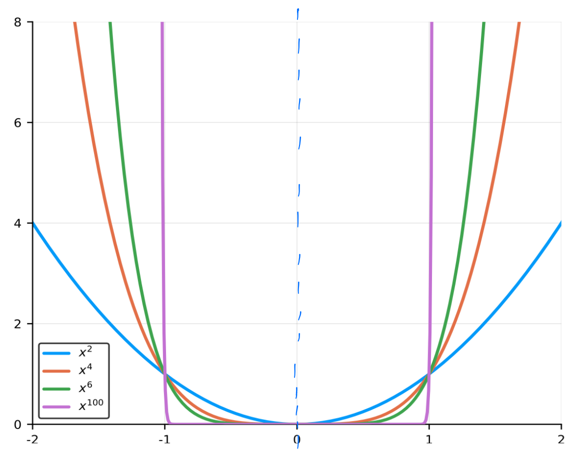
A function f is called even if $f(x) = f(-x)$

odd if $f(x) = -f(-x)$

Examples

$f(x) = x^n$ is even when n is even

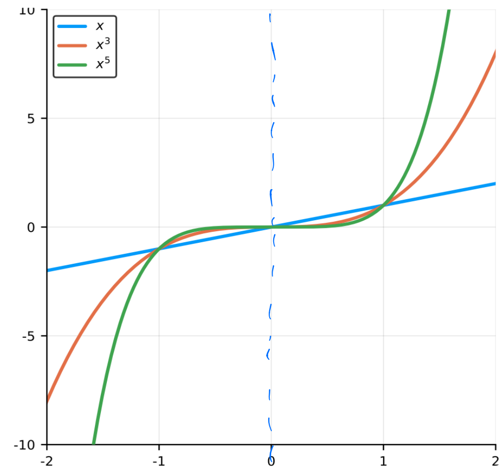
odd when n is odd



even functions

$f(x) = f(-x)$

symmetric

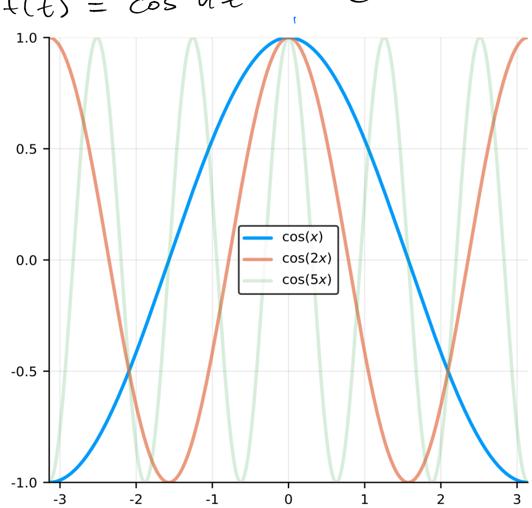


odd functions

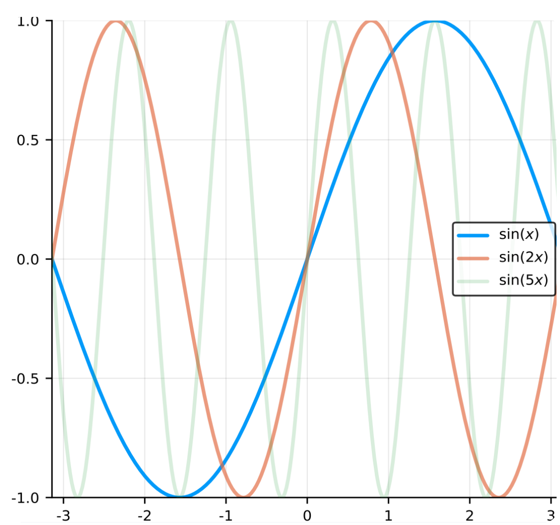
$f(x) = -f(-x)$

antisymmetric

$f(t) = \cos nt$ even



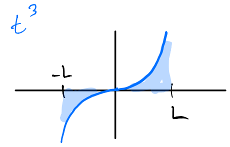
$f(t) = \sin nt$ odd



Some useful properties of odd and even functions

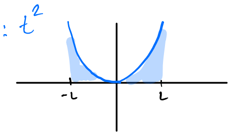
f odd : $\displaystyle\int_{-L}^{L} f(t)\,dt = 0$

eg: t^3



f even : $\displaystyle\int_{-L}^{L} f(t)\,dt = 2\int_{0}^{L} f(t)\,dt$

eg: t^2



odd + odd = odd — even + even = even

(even)(even) = even — (odd)$\cdot$(odd) = even ?

(odd)$\cdot$(even) = odd — (even)$\cdot$(odd) = odd

how to remember these:

even function behaves like positive number

odd " " " negative "

example proof: let f odd, g odd, and define

$h(t) = f(t)\,g(t)$ $\qquad h(-t) \stackrel{?}{=} h(t)$

$$\begin{aligned} h(-t) &= f(-t)\,g(-t) \\ &= (-f(t))\,(-g(t)) \qquad \text{because } f, g \text{ are odd} \\ &= f(t)\,g(t) \end{aligned}$$

$$\left[\begin{aligned} f(t) &= -f(-t) \\ -f(t) &= f(-t) \end{aligned}\right]$$

$h(-t) = h(t)$

i.e. h is even.

Half range expansions

Suppose $f : [0, L] \to \mathbb{R}$

the even expansion of f is the map g defined by

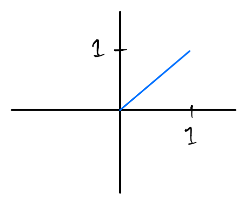
$$g(t) = \begin{cases} f(t) & \text{for } 0 \leq t \leq L \\ f(-t) & \text{for } -L \leq t < 0 \end{cases}$$

the odd expansion of f is

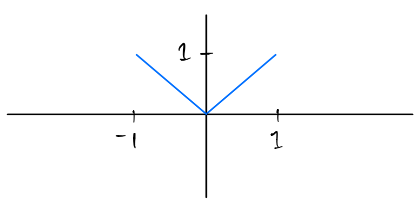
$$h(t) = \begin{cases} f(t) & \text{for } 0 < t \leq L \\ 0 & \text{at } t = 0 \\ -f(-t) & \text{for } -L \leq t < 0 \end{cases}$$

Example

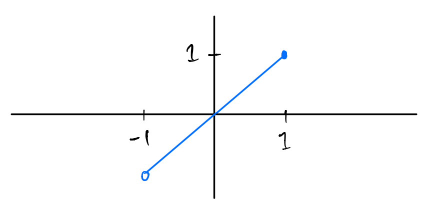
$f : [0,1] \to \mathbb{R}, \quad f(t) = t$



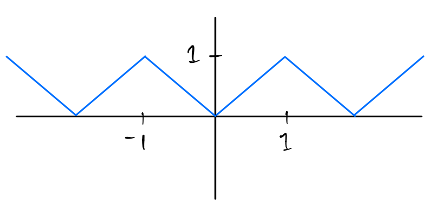
even expansion

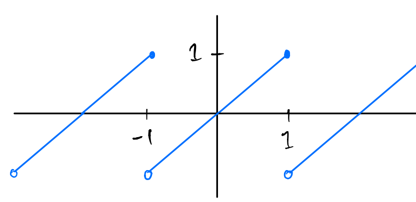


odd expansion



periodic
extensions





Example

Find the Fourier coefficients for the even expansion of $f(t)=t$ on $[0,1]$.

the expansion is

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ -t & -1 < t < 0 \end{cases}$$

$$a_0 = \int_{-1}^{1} f(t)\,dt = 2\int_0^1 t\,dt \qquad \text{because } f \text{ is even}$$

$$= 2\left[\frac{t^2}{2}\right]_0^1 = 1$$

$$\int uv' = uv - \int u'v$$

$$a_n = \int_{-1}^{1} \underbrace{f(t)}_{\text{even}} \underbrace{\cos(n\pi t)}_{\text{even}}\,dt = 2\int_0^1 \underbrace{t}_{u} \underbrace{\cos(n\pi t)}_{v'}\,dt = 2\left[\cancel{t\,\frac{\sin(n\pi t)}{n\pi}}\right]_0^1 - 2\int_0^1 \frac{\sin(n\pi t)}{n\pi}\,dt$$

$\sin n\pi = 0$

even $\times$ even $=$ even

$$= 0 + 2\left[\frac{\cos(n\pi t)}{n^2\pi^2}\right]_0^1$$

$$= 2\frac{(\cos(n\pi) - 1)}{n^2\pi^2} = \frac{2}{n^2\pi^2}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{-4}{n^2\pi^2} & n \text{ odd} \end{cases}$$

$$b_n = \int_{-1}^{1} \underbrace{f(t)}_{\text{even}} \underbrace{\sin n\pi t}_{\text{odd}}\,dt = 0$$

even $\times$ odd $=$ odd

FS : $$\frac{1}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi t$$

Observation: we started with an even function, and the FS has only even terms (cos is even). THIS ALWAYS HAPPENS!

IN GENERAL:

- If $f(t)$ is even then the FS has only constant + cosine terms (all the b_n coefficents of sin are zero)
- If $f(t)$ is odd then the FS has only sine terms (all the a_n coefficients of cos are zero).

Parseval's theorem

$f(t)$ 2π-periodic, bounded and piecewise continuous on $[-\pi, \pi]$

with FS $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

then
$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(t)]^2 \, dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

We have previously seen that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent – Parseval's theorem can be used to find what it actually converges to (a bit like how we used Taylor series of $\ln(1+x)$ to find $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$)

Recall: FS of
$$f(t) = \begin{cases} 1, & -\pi < t \leq 0, \\ -1, & 0 < t \leq \pi, \end{cases} \quad \text{and} \quad f(t+2\pi) = f(t).$$

is
$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{\pi n} \sin nt$$

Parseval's theorem gives:
$$\frac{1}{\pi}\left(\int_{-\pi}^{0} (-1)^2 \, dt + \int_{0}^{\pi} 1^2 \, dt\right) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \left(\frac{-4}{\pi n}\right)^2$$

i.e.

$$2 = \frac{16}{\pi^2} \sum_{\substack{n=1 \\ \text{odd}}} \frac{1}{n^2}$$

$$\sum_{\substack{n=1 \\ n \text{ odd}}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

now

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{\text{odd}} \frac{1}{n^2} + \sum_{\text{even}} \frac{1}{n^2}$$

$$= \frac{\pi^2}{8} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

rearranging:

$$\frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$