Infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$

Partial sums $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

Series converges if the sequence of partial sums is a convergent sequence.

The geometric series, $r \in \mathbb{R}$ ← constant ratio between successive terms

$$\sum_{i=0}^{\infty} r^i = 1 + r + r^2 + \cdots$$

converges to $\frac{1}{1-r}$ iff $|r| < 1$.

The p-series $p \in \mathbb{R}$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges iff $p > 1$. Don't worry about what it converges to.

special case $p = 1$ is the **harmonic series**

The divergence test

If the sequence of terms a_n do not converge to zero then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The comparison test

Two series $\sum a_n$, $\sum b_n$ with $0 \leq a_n \leq b_n$ (eventually)

- If $\sum b_n$ is convergent then $\sum a_n$ is convergent

  $\sum a_n$ gets squeezed

- If $\sum a_n$ is divergent then $\sum b_n$ is divergent

  $\sum b_n$ gets carried away

Limit comparison test

Two series $\sum a_n$, $\sum b_n$, $a_n \geq 0$, $b_n \geq 0$ (eventually)

Let $c = \lim\limits_{n \to \infty} \dfrac{a_n}{b_n}$

- If $0 < c < \infty$ then $\sum a_n$ convergent $\Leftrightarrow$ $\sum b_n$ convergent
- If $c = 0$ then $\sum b_n$ convergent $\Rightarrow$ $\sum a_n$ convergent
- If $\dfrac{a_n}{b_n} \to \infty$ then $\sum a_n$ convergent $\Rightarrow$ $\sum b_n$ convergent.

The integral test

$\sum a_n$ series with $a_n \geq 0$, f a function such that $f(n) = a_n$

If f is positive $f(x) > 0$, decreasing and continuous then

$$\sum_{n=1}^{a_n} a_n \text{ converges} \Leftrightarrow \int_1^\infty f(x)\,dx \text{ converges}$$

(used to prove convergence conditions for p-series)

Alternating series test

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

If $a_n \geq 0$, $a_n \geq a_{n+1}$ (eventually), and $\lim\limits_{n \to \infty} a_n = 0$

then $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ converges

$\sum a_n$ is called **absolutely convergent** if $\sum |a_n|$ converges

absolutely convergent $\Rightarrow$ convergent

the converse ($\Leftarrow$) does not hold

If $\sum a_n$ is convergent but $\sum |a_n|$ is not then $\sum a_n$ is called **conditionally convergent.**

The ratio test

$\sum a_n$ such that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \in \mathbb{R}$

absolute ratio between successive terms approaches a constant: compare geometric series

if $L < 1$ then $\sum a_n$ is absolutely convergent

if $L > 1$ then $\sum a_n$ is divergent

if $L = 1$ inconclusive

Power series centred at $c \in \mathbb{R}$. powers of $(x-c)$, $x \in \mathbb{R}$ too

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n + \dots$$

convergence depends on a_n, x, c according to ratio test

can be used to express/approximate a function $f(x)$ as a series of powers of $(x-c)$.

special case:

Taylor series centred at c. f differentiable

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

when convergent.